Online Appendix B:

Equilibrium Selection via the Lowest Continuation Surplus Refinement and its Microfoundation via Preference Shocks

Contents

B.1 Refinement Idea and Intuition

In the main text, we establish equilibrium uniqueness when the clients' beliefs are weakly increasing in initial types either globally, as shown in Observation 2 or only during quiet periods as shown in Observation A.1. This restriction on the off-path belief process $(k_t)_{t\geq 0}$ is intuitive and clear, so we adopt it in the paper's main text. In this Online Appendix B, we show that this same equilibrium is uniquely selected from a limiting perturbation approach to off-equilibrium beliefs. This approach is related to equilibrium selection in Quantal Response Equilibria (QRE) of McKelvey and Palfrey (1995) and is equivalent to the divinity refinement of Cho and Kreps (1987) whenever the continuation value of the players after separation is independent of the private information. In what follows, we describe our perturbation approach, characterize the resulting equilibrium refinement, and then go on to prove uniqueness under this refinement. We then consider a two-type model and construct the equilibria of the perturbed model explicitly, highlighting the existence of the limiting sequence in the simplified setting.

Challenges of known refinements. The key challenge in pinning down equilibrium beliefs is in understanding the clients' beliefs during quiet periods when there is no separation on-path. Existing signaling models feature a single informed player, which allows the application of known equilibrium refinements such as divinity. Such refinements consider which types of agents are most likely to deviate and assign the most off-path likelihood to those agents. Our model is distinct in that there are two informed players – an agent and an intermediary – with distinct preferences. Application of known refinements requires attribution of which of the players initiates the split. Even then, we show in Section B.7 that if a divine equilibrium exists, either under a reasonable extension to a three-player model or the standard definition of D1 to a two-player version of our model, then it coincides with the quiet-churning equilibrium we construct in Section 3 of the main text. We prove that a divine equilibrium exists if the quiet period is not too long, as proxied by a not too small intermediary outside option V. Surprisingly, however, we show that if the intermediary's outside option V is too low, then a divine equilibrium may not exist at all. This is a surprising finding and we present a detailed analysis, discussion, and illustrations in Section B.7 of this Online Appendix.

These challenges make known refinements inapplicable directly, and for this reason, we approach the model from first principles by introducing disutility shocks akin to Acemoglu and Pischke (1998) into the employment relationship between the intermediary and the agent and studying the belief refinement that arises as the variance of these shocks converges to zero.

Perturbation approach. We assume, similar to Acemoglu and Pischke (1998), that in every instance an agent can receive a discrete disutility shock for working with the intermediary.[1] This disutility shock has an exponential distribution with parameter $\Delta > 0$ and rate of arrival ε.[2] Upon the arrival of this shock, the intermediary and the agent choose whether the value of the continued relationship is sufficient for the intermediary to compensate for the agent's disutility shock.[3] Consequently, the agent is retained only if the continued value of the relationship exceeds the value of the shock. Such stochastic shocks imply that there do not exist quiet periods in the perturbed model as in every instance, there is a positive probability of every agent receiving a high enough shock to trigger a split. Tracking the evolving distribution of separating types becomes highly complex and it is generally infeasible to solve this perturbed model in closed form.[4] We show, however, that as the variance of the shocks in the perturbed model converges to 0, as captured by Δ increasing to infinity, the beliefs during any quiet period converge to the intermediary-agent pair that has the lowest equilibrium continuation surplus relative to the value of separating immediately. The resulting refinement is consistent with a stability intuition in that the players that have the most to lose from deviating would be least likely to do so. The resulting refinement echoes the relationships between divinity and stability, as shown by Cho and Sobel (1990), and coincides with those refinements in signaling games whether the ex-post value of signaling is independent of the agent's private beliefs, such as Spence (1973), Noldeke and Van Damme (1990), Swinkels (1999), Kremer and Skrzypacz (2007) among others. This refinement also presents an illustration how Quantal Response Equilibria of McKelvey and Palfrey (1995) can be used in equilibrium selection in signaling models. This equilibrium selection approach is formulated in Section B.2 and the limiting argument is presented in Section B.4.

Equilibrium uniqueness. We show in Section B.3 that the equilibrium constructed in Proposition 1 of the main text, featuring a quiet period followed by a churning period, is the unique equilibrium satisfying our refinement subject to an additional regularity condition that belief process $(k_t)_{t\geq 0}$ is continuous. First, we show that there must exist a final date $\bar{t}$ by which agents of all types leave the intermediary. If there is pooling prior to $\bar{t}$, then there must be a quiet period prior to $\bar{t}$. Our belief refinement then pins down that belief during this quiet period should equal to the belief of the lowest type separating at time $\bar{t}$. This results in a jump in beliefs at $\bar{t}$, which contradicts belief continuity. We then extend this argument to show that the equilibrium must be separating prior to $\bar{t}$, allowing us to pin down the dynamics of the

[1] Due to flexible wages, it is not important as to which party, the intermediary or the agent, are subject to the disutility shock.

[2] The argument extends to any distribution with a decreasing likelihood ratio property.

[3] This ex-post negotiation between the intermediary and the agent alleviates the need for clients to attribute deviations to a particular party since, if a deviation is observed, the clients simply infer that no mutual agreement has been reached.

[4] In Section B.6.2 we are able to make progress by constructing an equilibrium if the ex-ante asymmetric information $\tilde{p}_0$ is binary.

churning period. Finally, and just like in the equilibrium constructed in Section 3 of the paper, the lowest skilled agents have the least surplus when working for the intermediary, resulting in the clients' beliefs tracking this worst agent during the quiet period. For tractability, we conduct this analysis under the assumption of Section 3 of the main text that $A(\pi(p,t))$ is concave in t, but this analysis can be extended beyond it with additional work. This first principles approach provides a rigorous micro-foundation for the equilibrium we construct in Section 3.1.

Online Appendix B structure. In Section B.2 we introduce the necessary notation, propose the candidate off-path refinement, discuss its relationship with existing refinements, formulate the main result and provide the intuition for the proofs. We show formally in Section B.3 that the equilibrium we construct in Section 3 of the main text is unique under our proposed refinement and the additional regularity condition that client beliefs are continuous along the path of good performance. In Section B.4 we construct the perturbed version of the model and show that the limiting beliefs of the perturbed model as the variance of the shocks converges to zero has to satisfy the belief refinement outlined in Section B.2 and, consequently, must converge to the unique equilibrium derived in Section B.3. We extend our results to the binary version of the model in Section B.5, i.e., one in which the initial type $\tilde{p}_0$ features only two types $\left\{\underline{p},\overline{p}\right\} = \left\{p^L, p^H\right\}$ and explicitly construct an equilibrium of the perturbed version of the model in Section B.6. Finally, in Section B.7, we consider the divinity refinement in the context of the two type model and show that a divine equilibrium may not exist when the initial quiet period is long.

B.2 Lowest Continuation Surplus Belief Refinement

The model setting mimics the description in Section 3 of the main text. Additionally, we assume that the agent's outside option $L = 0$ which, as we have shown in the proof of Observation A.1, is without loss. For tractability, we will assume that the intermediary and the agent share the same discount rate r. Finally, as in Section 3 of the main text we assume that $A(\pi(\underline{p},t))$ is weakly concave in t. This is done for simplicity due to the already extensive nature of the proofs in this Online Appendix B.

To streamline the notation, throughout Online Appendix B, we denote by $p = \tilde{p}_0$ the ex-ante type of the agent as indexed by the value of his private signal at the start of the game. The baseline equilibrium notion follows the Perfect Bayesian Equilibrium definition 1 in Section 2.

As in the main text proofs, define by $l_t \overset{def}{=} \pi(k_t, -t)$ to be the ex-ante type that clients assign to an agent leaving the intermediary at time t along the path of good performance. Similarly, define by $m_t \overset{def}{=} \pi(q_t, -t)$ to be the ex-ante average type along the path of good performance. We have shown in Lemma A.16 of the Online Appendix A that it must be the case that $l_t \leq m_t$ in every equilibrium.

On-path separations. Denote by $\tau(p) \subset [0, +\infty]$ to be the, possibly random, on-path stopping time when type p leaves the intermediary along the path of good performance in a candidate equilibrium. Denote $\mathbb{T}(p) \stackrel{def}{=} \textit{support}(\tau(p))$ to be the support of $\tau(p)$ for each p. Denote the last date when type p separates in equilibrium by $\bar{t}(p) \stackrel{def}{=} \sup_{p \in [\underline{p}, \bar{p}]} \mathbb{T}(p)$. Denote by $\mathbb{T} \stackrel{def}{=} \textit{support}(\tau) = \cup_p \mathbb{T}(p)$ to be the set of times when an agent separates from the intermediary along the equilibrium path. Denote by $\bar{t} \stackrel{def}{=} \sup \mathbb{T} = \sup \bar{t}(p)$ to be the last time when a type separates in equilibrium. We have shown in Lemma A.17 of the Online Appendix A that $\bar{t}$ is finite and bounded by (A.86). Denote by $R(t)$ the support of ex-ante types that stay with the intermediary beyond time t with a positive probability along the equilibrium path

$$R(t) \stackrel{def}{=} \textit{closure}\big\{p : \quad \mathrm{P}_p(\tau(p) > t) > 0\big\}. \tag{B.1}$$

Continuation values. Denote by $W_t(p)$ to be the joint continuation welfare at time t of the intermediary and the agent with ex-ante skill p who has performed well up to time t is given by:

$$\begin{aligned} W_t(p) \stackrel{def}{=} \max_{\hat{\tau} \geq t} \bigg\{ \int_t^{\hat{\tau}} & e^{-r(s-t)} \cdot \Big[\pi(p,t) + (1 - \pi(p,t)) \cdot e^{-\lambda(s-t)}\Big] \cdot \Big[A(\pi(m_s, s)) - rV\Big]\, ds \\ & + e^{-r(\hat{\tau}-t)} \cdot \Big[\pi(p,\hat{\tau}) \cdot u_1(\pi(l_{\hat{\tau}}, \hat{\tau})) + (1 - \pi(p,\hat{\tau})) \cdot e^{-\lambda(\hat{\tau}-t)} \cdot u_0(\pi(l_{\hat{\tau}}, \hat{\tau}))\Big] + V \bigg\}. \end{aligned} \tag{B.2}$$

If $t \leq \bar{t}(p)$, then $W_t(p)$ is the on-path continuation value, while if $t > \bar{t}(p)$, then $W_t(p)$ is the off-path continuation value for the agent of ex-ante type p. The outside option at time t of the agent of ex-ante type p who has performed well up to time t, given client belief l_t, is given by

$$U_t(p) \stackrel{def}{=} U(\pi(p,t), \pi(l_t,t)) = \pi(p,t) \cdot u_1(\pi(l_t,t)) + (1 - \pi(p,t)) \cdot u_0(\pi(l_t,t)). \tag{B.3}$$

Denote by $V_t(p)$ to be the continuation value at time t of the intermediary employing the agent of ex-ante type p who has performed well up until time t:

$$V_t(p) \stackrel{def}{=} W_t(p) - U_t(p). \tag{B.4}$$

The value added of the agent working for the intermediary is equal to the gain $V_t(p) - V$ that the intermediary gains relative to her outside option. This value also captures the joint equilibrium surplus of the intermediary-agent pair relative to the joint value of their outside options.

B.2.1 Lowest Continuation Surplus Belief Refinement Definition

We can now formally introduce the off-path belief refinement in which the clients attribute off-path separations to intermediary-agent types p that have the lowest continuation surplus $W_t(p)$ relative to their joint outside option $U_t(p) + V$. Following definition (B.4), this joint surplus is equal to $V_t(p) - V$, i.e., the intermediary's expected surplus from employing the agent of ex-ante type p from time t onwards. This is motivated by the idea that, if the intermediary and the agent were to receive relationship specific shocks, then it is precisely the pairs with the lowest continuation surplus who would be most likely to terminate their relationship.

Definition 1 (Lowest continuation surplus belief refinement)**.** *An equilibrium satisfies the lowest continuation surplus" refinement if for every $t \in [0, \bar{t}] \setminus \mathbb{T}$ the off-equilibrium path belief l_t lies in the convex hull of the support of the residual types $R(t)$, i.e., $l_t \in [\min R(t), \max R(t)]$ and corresponds to the intermediary-agent pair with the lowest continuation surplus*

$$V_t(l_t) = \min_{x \in [R(t)]} V_t(x). \tag{B.5}$$

Definition (1) requires, first, that clients assign off-path deviations to types that remain with the intermediary with positive probability at time t along the path of good performance. This first condition is equivalent to condition (ii) of Observation (ii) and can be equivalently restated that clients place zero weight on agents who have separated from the intermediary with certainty. The intuition for it is that only agents who are still employed by the intermediary would be subjected to separation shocks – a result we derive formally in Section B.4 when considering the perturbed version of the model. Second, among the ex-ante types that remain with the intermediary, clients assign off-path deviations to those who have the lowest continuation value surplus. Requirement (B.5) is also motivated by the perturbed version of the model and we derive its necessity in Section B.4 when considering the effect of relationship-specific shocks. While we require that definition 1 applies only to off-path beliefs, it is clear that on-path separations must also satisfy both of the outlined requirements following Bayesian consistency of client beliefs (3) and separation optimality in (6).

The definition of $V_t(p)$ in (B.4) captures the surplus obtained by the informed parties and, consequently, definition 1 can be extended beyond two informed players. Definition 1 is also less restrictive than requiring that $l_t \in \arg\min_{x \in R(t)} V_t(x)$ as it allows for beliefs l_t to lie in the convex hull of $R(t)$, thus allowing for mixed strategies, and such that $V_t(l_t)$ minimizes (B.5). It is also possible to embed right continuity of the belief process $(l_t)_{t \geq 0}$ in definition 1 by replacing (B.5) with $V_{t+}(l_t) = \min_{x \in R(t)} V_{t+}(x)$.

B.2.2 Relationship to Divinity and the Intuitive Criterion

Definition 1, as we mentioned above, can be extended to a single informed player, in which $V_t(p)$ denotes the equilibrium gain of the informed player. In this section we describe how the resulting refinement compares to established approaches in signaling games, such as the intuitive criterion of Cho and Kreps

(1987) and divinity in Cho and Sobel (1990), which is equivalent to universal divinity and independence of never-weak-best responses of Kohlberg and Mertens (1986) in our monotonic signaling game. We also relate Definition 1 to trembling hand perfection of Selten and Bielefeld (1988).

Comparison to trembling hand perfection. We start by highlighting that definition 1 is distinct from the trembling hand perfection of Selten and Bielefeld (1988). The lowest continuation surplus refinement assumes that an off-path deviation is much more likely to arise from a type that has the lowest continuation surplus, whereas trembling hand perfection assumes that the probability of a deviation is uniform across private types. The latter assumption implies that off-path beliefs must reflect the average player type that has not separated with positive probability, which is economically restrictive as it immediately rules out separating equilibria in numerous signaling models such as Spence (1973) and Noldeke and Van Damme (1990) among others.

We show in Section B.6 that the lowest continuation surplus refinement follows from a limiting perturbation argument in which the informed party receives a stochastic shock. Such perturbations result in a Quantal Response Equilibrium (QRE) as pioneered by McKelvey and Palfrey (1995) and McKelvey and Palfrey (1998) who also show that the selected equilibria differ from the trembling hand perfection of Selten and Bielefeld (1988) as the probabilities of off-path deviations are not uniformly distributed across informed player types and, instead, decrease in their continuation payoff, just like in definition 1.

Comparison to divinity. The expected value $W_t(p)$ of the intermediary-agent pair, defined in (B.2), is increasing in the clients' belief process l. Cho and Sobel (1990) show that such monotonicity results in an equivalence between universal divinity and independence of never-weak-best-responses. Consequently, the comparison of our refinement to divinity also speaks to these two established refinements.

The divinity refinement attributes off-path separations to a private type p that is willing to separate off-path for the widest range of hypothetical beliefs by investors. In the context of the monotone signaling game like hours, for a given type p we can define a time t indifference belief $d_t(p)$ such that the equilibrium payoff $W_t(p)$ to the informed player, is equal to immediate separation at the client belief $d_t(p)$. Formally, $d_t(p)$ is a solution to

$$W_t(p) = V + U(\pi(p,t), \pi(d_t(p), t)). \tag{B.6}$$

Due to the monotonicity of the signaling game, player p is then willing to separate off-path for any belief $(d_t(p), 1]$ assigned to him by clients. The divinity refinement then attributes an off-path separation to a player type p with the broadest range of client beliefs $(d_t(p), 1]$ for which he is willing to separate, which can be rewritten as the clients' belief l_t to belong to $\arg\min_p d_t(p)$. The divinity refinement, thus, considers informed player indifference in the client belief space. The lowest continuation surplus refinement defined via Definition 1 focuses on expected equilibrium values and explores which types are more likely to deviate in response to match preference shocks. This is conceptually similar, but can and should be thought of as stability-like perturbations to the informed agent's expected payoff space, rather than perturbations to client beliefs.

Definition 1 is equivalent to divinity whenever the expected post-separation value of the informed player is independent from his private belief, e.g., if the right hand side of (1) is independent of the private belief p. This is satisfied in signaling games such as Spence (1973) and Kremer and Skrzypacz (2007) where the agent's expected value post separation is independent of future information flow. In this case, the indifference belief $d_t(p)$ is strictly increasing in the agent's expected value $W_t(p)$ and, consequently, $\arg\min_p W_t(p) = \arg\min_p d_t(p)$. Consequently, the distinction between divinity and the lowest continuation surplus refinement depends on the dependence of the agent's post-signaling value on his type. We show in Section B.7.2.1 that this is a meaningful distinction by illustrating the existence of a lowest continuation surplus equilibrium and non-existence of the corresponding divine equilibrium.

Comparison to the intuitive criterion. In a monotone signaling game such as ours, the intuitive criterion of Cho and Kreps (1987) rules out a time t deviation by player of ex-ante private type p if the equilibrium expected payoff $W_0(p)$ by this player is greater than the expected payoff this player would have received if he was perceived as the highest skilled type $\bar{p}$ at the time of his deviation. The intuitive criterion leads to a unique equilibrium in the two-type model of Spence (1973), but results in multiple equilibria when there are more than two types. A similar limitation arises in our setting – the equilibrium we construct in Section 3 of the main text survives the intuitive criterion, however the intuitive criterion does not discipline beliefs during any and every quiet period as every agent would prefer to be separate while being perceived as the highest type. The more formal distinction is that definition 1 considers the types that are most likely to deviate, which we micro-found in Section B.4 via a stochastic perturbation of the model, while the intuitive criterion rules out a rather narrow set of types that would be highly unlikely to deviate which is insufficient to refine the set of equilibria when there are either more than two types, or when there is a quiet period. We can show that when there are just two types, then the intuitive criterion results in the same equilibrium as the lowest continuation surplus refinement in 1.

B.2.3 Unique Equilibrium Satisfying Lowest Continuation Surplus Refinement

We now state the main result of this Online Appendix B – the equilibrium we construct in Section 3 is unique under the lowest continuation surplus refinement 1.

Proposition B.1 (Unique Equilibrium). *Suppose $A(\pi(p,t))$ is concave in time t and $L = 0$. The Perfect Bayesian Equilibrium constructed in Proposition 1 is the unique equilibrium in which client belief process l is continuous along the path of good performance and satisfies the lowest continuation surplus refinement*

1.

Given the extensive nature of the proofs, we provide these results under the simplifying assumption of Section 3 that $A(\pi(p,t))$ is concave in t. As we have shown in the proof of Observation A.1, it is without loss to assume that $L = 0$ and we do not repeat the argument here out of concision. We also assume that the client belief process l is continuous – as we show in the formal proofs of Section B.3 this eliminates pooling in the final period without relying on any additional refinement assumptions. We overview the intuition and steps for the proof of Proposition B.1 written out formally in Section B.3.

First, as shown by Lemma A.17, the equilibrium separations must happen in finite time. We then work backwards. If there is pooling in the final period $\bar{t} = \mathit{support}(\tau)$, we show that there must be a quiet period preceding it. As higher skilled agents have more to gain from reputation building, the lowest continuation surplus refinement 1 implies that client beliefs during the quiet period track the lowest skilled agent who separates at time $\bar{t}$. This, however, results in a jump in client beliefs at time $\bar{t}$, implying that there cannot be pooling at time $\bar{t}$ – this is the only moment where belief continuity matters and we show that by relaxing this condition, the equilibrium is very similar, but may feature pooling in the final period.[5] Next, we show that separation prior to $\bar{t}$ must be gradual with no pools of agents separating from the intermediary – we show that such jumps would result in positive belief jumps which would violate incentive compatibility. We then apply the equivalent of Lemma A.18 which shows that gradual separation periods must be increasing in types and unique in the sense that there cannot be two distinct periods during which an agent would be willing to separate. These steps prove that there must be a unique churning period characterized by Proposition 1. Finally, as higher skilled agents have the most to gain from reputation building, the lowest continuation surplus refinement picks the lowest skilled agent as the most likely to deviate during the quiet period, thus completing the equilibrium characterization. We now proceed to the formal proofs.

B.3 Proof of Proposition B.1 (unique equilibrium)

Denote the set of times when it is weakly optimal for ex-ante type p to separate

$$\mathbb{T}^I(p) \overset{def}{=} \arg\max_{\tau} \Big\{ \int_0^{\tau} e^{-rt} \cdot \Big[p + (1-p) \cdot e^{-\lambda t}\Big] \cdot \Big[A(\pi(m_t, t)) - rV\Big]\, dt \\ + e^{-r\tau} \cdot \Big[p \cdot u_1(\pi(l_\tau, \tau)) + (1-p) \cdot e^{-\lambda\tau} \cdot u_0(\pi(l_\tau, \tau))\Big] + V \Big\}, \tag{B.7}$$

[5] Such final period separations can also be taken care of by expanding definition 1 to off-path separations that occur after time $\bar{t}$.

where superscript I in $\mathbb{T}^I$ stands for indifference. Denote by $\mathbb{T}^I \stackrel{def}{=} \cup_p \mathbb{T}^I(p)$ to be the set of times when there is an ex-ante type that finds it weakly optimal to separate.

Denote by $S(t)$ to be the set of ex-ante types which separate at time t with a positive probability, i.e.,

$$S(t) \stackrel{def}{=} \{p : \quad t \in \mathbb{T}(p)\} \qquad \textit{(on-path separation)} \tag{B.8}$$

Denote by $S^I(t)$ the set of ex-ante types for whom it is weakly optimal to separate at time t, i.e.,

$$S^I(t) \stackrel{def}{=} \{p : \quad t \in \mathbb{T}^I(p)\} \qquad \textit{(weak optimality)} \tag{B.9}$$

Define function $W_0(p;t)$ as the joint value obtained by type p for stopping at time t, given by

$$\begin{aligned} W_0(p;t) \stackrel{def}{=} & \int_0^t e^{-rs} \cdot \Big[p + (1-p) \cdot e^{-\lambda s}\Big] \cdot \Big[A(\pi(m_s, s)) - rV\Big]\, ds \\ & + e^{-rt} \Big[p \cdot u_1(\pi(l_t, t)) + (1-p)e^{-\lambda t} \cdot u_0(\pi(l_t, t))\Big] + V. \end{aligned} \tag{B.10}$$

As can be seen in (B.10), function $W_0(p;t)$ is linear in p.

Lemma B.1 (Separating set properties)**.** *Set $S^I(t)$ is convex for every t. If $\left|S^I(t_1) \cap S^I(t_2)\right| > 1$ then $S^I(t_1) = S^I(t_2)$ and $\mathbb{T}^I(p') = \mathbb{T}^I(p'')$ for every $p', p'' \in S^I(t_1) = S^I(t_2)$.*

Proof. Suppose $p_1, p_2 \in S^I(t)$. This implies that $t \in \mathbb{T}^I(p_1) \cap \mathbb{T}^I(p_2)$. It follows that $W_0(p_1) = W_0(p_1;t)$ and $W_0(p_2) = W_0(p_2;t)$. Due to weak convexity of $W_0(p)$, it implies that $W_0(p) = W_0(p;t)$ for all $p \in [p_1, p_2]$. Consequently, $S^I(t)$ is a convex set.

Following the above argument, $W_0(p)$ is linear over any $(p_1, p_2) \subseteq S^I(t)$ for each t. This implies that for any $p_1 \in S^I(t_1)$ and $p_2 \in S^I(t_2)$, and $\hat{p} \in S^I(t_1) \cap S^I(t_2)$ it follows that $W_0'(p_1;t) = W_0'(\hat{p};t) = W_0'(p_2;t)$. Since $W_0(p_1;t) = W_0(p_1)$ and $W_0(p_2;t) = W_0(p_2)$, then it follows that $W_0(p)$ is linear over $[p_1, p_2]$ and equal to $W_0(p;t)$. □

Lemma B.2 (Stopping time properties)**.** *Suppose $p_1, p_2 \in S^I(t)$. Then $\mathbb{T}^I(p_1) = \mathbb{T}^I(p_2) = \mathbb{T}^I(\hat{p})$ for any $\hat{p} \in [p_1, p_2]$.*

Proof. From Lemma B.1 it follows that $[p_1, p_2] \in S^I(t)$, meaning that $W_0(p)$ is linear over $[p_1, p_2]$ and $W_0(p) = W_0(p;t)$ for every $p \in [p_1, p_2]$. Consider a type $p' \in (p_1, p_2)$ and suppose that $t' \neq t$ and $t' \in \mathbb{T}^I(p')$. This implies that $W_0(p') = W_0(p';t')$. Then it must be the case that

$$W_0'(p') = \frac{\partial}{\partial p} W_0(p;t')\Big|_{p=p'}. \tag{B.11}$$

If (B.11) did not hold, then $W_0(p'-\varepsilon) < W_0(p'-\varepsilon;t')$ or $W_0(p'+\varepsilon) < W_0'(p'+\varepsilon;t')$, which would contradict with the definition of $W_0(p)$. □

Lemma B.3 (Strict convexity of $W_0(p)$)**.** *Value function $W_0(p)$ is strictly convex if and only if $|S^I(t)| \in \{0,1\}$ for each t.*

Proof. Suppose $W_0(p)$ is strictly convex. Then it implies that there exists at most a single type such that $W_0(p) = W_0(p;t(p))$ given the tangency condition. Conversely, if $|S^I(t)| > 1$ then Lemma B.1 implies that $W_0(p) = W_0(p;t)$ is linear over $S^I(t)$, contradicting strict convexity of $W_0(p)$. □

Lemma B.4 (Best type separates last)**.** *In any equilibrium it must be the case that $\bar{t} \in \mathbb{T}^I(\bar{p})$, i.e., the highest ex-ante type $\bar{p}$ finds it weakly optimal to separate at the final date $\bar{t}$.*

Proof. Suppose $\bar{t}(\bar{p}) < \bar{t}$. Denote $\hat{l} \overset{def}{=} \inf\left\{S\left(\mathbb{T}^I(\bar{p})\right)\right\}$ to be the lowest type that is willing to separate at the same time as $\bar{p}$. Following Lemma B.1, $\hat{l}$ is well defined and $S^I(\mathbb{T}^I(\bar{p})) = \left[\hat{l},\bar{p}\right]$. For $t > \bar{t}(\bar{p})$ it follows that the only types that separate are types below $\hat{l}$, implying that $l_{\bar{t}(\bar{p})} \geq m_{\bar{t}(\bar{p})}$, which contradicts Lemma A.16 used in the proof of Observation A.1. Hence, it must be the case that $\bar{t} \in \mathbb{T}^I(\bar{p})$. □

Suppose process $(l_t)_{t\geq 0}$ is differentiable at t. It follows from (A.12) in the proof of Proposition 2 that the intermediary's continuation value $V_t(p)$ is also differentiable at t and satisfies

$$\begin{aligned} rV_t(p) - \dot{V}_t(p) = {} & A(\pi(m_t,t)) - A(\pi(l_t,t)) - rV \\ & + \left[\pi(p,t)\cdot u_1'(\pi(l_t,t)) + (1-\pi(p,t))\cdot u_0'(\pi(l_t,t))\right]\cdot \partial_1\pi(l_t,t)\cdot \dot{l}_t. \end{aligned} \tag{B.12}$$

Equilibria Satisfying the Lowest Surplus Belief Refinement 1

Npw we identify the set of equilibria that satisfy the lowest continuation surplus refinement 1. Define $t_S \overset{def}{=} \sup\{t < \bar{t}:\ t \in \mathbb{T}\}$ to be the last moment in which an agent leaves the intermediary along the equilibrium path prior to the final date $\bar{t}$.

Lemma B.5 (Final pooling properties)**.** *Suppose $|S(\bar{t})| > 1$. Then, if the equilibrium satisfies the lowest continuation surplus refinement 1, then $t_S < \bar{t}$. If, additionally, belief process $(l_t)_{t\geq 0}$ is right-continuous, then $l_t = \min R(t) = \min S(\bar{t})$ for every $t \in [t_S, \bar{t})$.*

Proof. **Quiet period.** Suppose $t_S = \bar{t}$, meaning that there exists a sequence of times $(t_n)_{n\geq 0} \in \mathbb{T}$ and such that $t_n \to \bar{t}$. Since $l_{\bar{t}} = m_{\bar{t}}$, incentive compatibility of the separating types requires that $\lim_{n\to\infty} l_{t_n} = m_{\bar{t}}$.

Suppose $S^I(t_{n_j}) \neq S^I(\bar{t})$ for some sub-sequence $(t_{n_j})_{j\in\mathbb{N}} \subseteq (t_n)_{n\in\mathbb{N}}$. Following Lemmas B.1 and B.4, set $S^I(\bar{t})$ is convex and includes $\bar{p}$, it implies that $l_{t_{n_j}} \in \left[\underline{p}, \min S(\bar{t})\right]$. Since $|S(\bar{t})| > 2$ it follows that

$m_{\bar{t}} > \min S(\bar{t})$, implying a contradiction with $\lim_{n\to\infty} l_{t_n} = m_{\bar{t}}$. Hence it must be the case that there exists $N > 0$ such that $S^I(t_n) = S^I(\bar{t})$ for $n \geq N$.

Suppose $S^I(t_n) = S^I(\bar{t})$ for every $n \geq N$ implying that the intermediary letting go of agent of skill $p \in S(t_n)$ at time t_n is weakly indifferent to waiting until time $\bar{t}$ to let that agent go. For each $p_0 \in S^I(\bar{t})$ define by $L_t(p)$ to be a hypothetical indifference belief such that the joint value to the intermediary and the agent at time t is equal to the expected value of waiting until time $\bar{t}$ and be perceived as the posterior type $l_{\bar{t}} = m_{\bar{t}}$. Formally, $L_t(p)$ solves

$$V + U_t(L_t(p)) \stackrel{def}{=} W_t(p). \tag{B.13}$$

Belief process $L_t(p)$ is specific to the ex-ante type p. However, by continuity, it follows that $L_{\bar{t}}(p) = l_{\bar{t}} = m_{\bar{t}}$ for every $p \in S^I(\bar{t})$. Moreover, by the previous observation that $S^I(t_n) = S^I(\bar{t})$, it follows that $L_{t_n}(p) = l_{t_n}$ for every $p \in S^I(\bar{t})$ and $n \geq N$. Differentiating (B.13) with respect to t obtain that

$$\dot{L}_t(p) = \frac{A(\pi(L_t(p), t)) + rV - A(\pi(m_t, t))}{\partial_1 \pi(L_t(p), t) \, \cdot \, \partial_2 U(\pi(p, t), \pi(L_t(p), t))}. \tag{B.14}$$

Since $m_{\bar{t}} = l_{\bar{t}} = L_{\bar{t}}(p)$ for every $p \in S^I(\bar{t})$, it follows that $\frac{\partial}{\partial p} \dot{L}_{\bar{t}}(p) < 0$. This implies that there exists $\varepsilon > 0$ such that $\dot{L}_t(p)$ is strictly decreasing in p for every $t \in (\bar{t} - \varepsilon, \bar{t})$. This, in turn, implies that $L_t(p)$ is strictly increasing in p_0 for every $t \in (\bar{t} - \varepsilon, \bar{t})$, leading to a contradiction with the fact that $L_{t_n}(p) = l_t$ for every $p \in S^I(t_n)$ and $n \geq N$. This proves that if $|S(\bar{t})| > 1$, then there exists a $t_S < \bar{t}$ such that $(t_S, \bar{t}) \cap \mathbb{T} = \emptyset$.

Off-path beliefs. Suppose there exists a $B_\epsilon(t_0) \in (t_S, \bar{t})$ such that $l_t \neq \inf S(\bar{t})$. Following Lemma A.16 it follows that $l_t \in (\inf S(\bar{t}), \sup S(\bar{t}))$. To satisfy the lowest continuation surplus refinement 1, the off-path belief l_t solves

$$V_t(l_t) = \min_{x \in R(t)} V_t(x) = \min_{x \in S(\bar{t})} V_t(x). \tag{B.15}$$

All types $p \in S^I(\bar{t})$ find it weakly optimal to wait until time $\bar{t}$ to separate from the intermediary. This implies that function $V_t(p)$ is linear in $\pi(p, t)$ for $p \in S^I(\bar{t})$. As $\pi(p, t)$ is increasing in p, it implies that the solution to (B.15) is interior if and only if $V_t'(p) = 0$ for each $p \in S^I(\bar{t})$ and $t \in B_\epsilon(t_0)$. Differentiating the definition of $V_t(p)$ in (B.4) obtain that $V_t'(p) = 0$ for $p \in S^I(\bar{t})$ if and only if l_t solves

$$\begin{aligned} u_1(\pi(l_t, t)) - u_0(\pi(l_t, t)) = &\int_t^{\bar{t}} \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot \Big(A(\pi(m_{\bar{t}}, s)) - rV\Big) \, ds \\ &+ \int_{\bar{t}}^{\infty} \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot A(\pi(m_{\bar{t}}, s)) \, ds. \end{aligned} \tag{B.16}$$

The solution l_t to (B.16) is differentiable for $t \in B_\epsilon(t_0)$. Note that $V_t'(p) = 0$ for all $t \in B_\epsilon(t_0)$ requires that $\dot{V}_t'(p) = 0$. Differentiating (B.12) with respect to p, obtain

$$0 = \partial_1\pi(p,t) \cdot \Big[u_1'(\pi(l_t,t)) - u_0'(\pi(l_t,t))\Big] \cdot \partial_1\pi(l_t,t) \cdot \dot{l}_t \qquad \Leftrightarrow \qquad \dot{l}_t = 0$$

for every $t \in B_\epsilon(t_0)$, implying that $l_t = l_{t_0}$ for every $t \in B_\epsilon(t_0)$. Using this, rewrite (B.16) as

$$rV \cdot \int_t^{\bar{t}} \left(e^{-rs} - e^{-rs-\lambda(s-t)}\right) ds = \int_t^\infty \left(e^{-rs} - e^{-rs-\lambda(s-t)}\right) \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) ds \qquad \text{(B.17)}$$

Equality (B.17) holds for every $t \in B_\epsilon(t_0)$ implying that the derivatives with respect to t of the left and right hand sides must be equal:

$$\begin{aligned} -r\lambda V \cdot \int_t^{\bar{t}} e^{-(r+\lambda)(s-t)} \, ds &= -\lambda \int_t^\infty e^{-rs-\lambda(s-t)} \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) \, ds \\ rV \cdot \int_t^{\bar{t}} e^{-(r+\lambda)(s-t)} \, ds &= \int_t^\infty e^{-rs-\lambda(s-t)} \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) \, ds. \\ rV \cdot \int_t^{\bar{t}} e^{-(r+\lambda)s} \, ds &= \int_t^\infty e^{-(r+\lambda)s} \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) \, ds. \end{aligned} \qquad \text{(B.18)}$$

Conditions (B.17) and (B.18) are equivalent to

$$\begin{cases} rV \cdot \int_t^{\bar{t}} e^{-rs} \, ds = \int_t^\infty e^{-rs} \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) \, ds, \\ rV \cdot \int_t^{\bar{t}} e^{-(r+\lambda)s} \, ds = \int_t^\infty e^{-(r+\lambda)s} \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) \, ds \end{cases} \qquad \text{(B.19)}$$

Conditions (B.19) must be satisfied for all $t \in B_\epsilon(t_0)$, which requires that

$$A(\pi(m_{\bar{t}},t)) - A(\pi(l_{t_0},t)) - rV = 0 \qquad \forall t \in B_\epsilon(t_0). \qquad \text{(B.20)}$$

If (B.20) is satisfied, then the indifference condition (B.16) requires that

$$\int_{\bar{t}}^\infty \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot \Big(A(\pi(m_{\bar{t}},s)) - A(\pi(l_{t_0},s))\Big) = 0,$$

which contradicts $l_{t_0} < m_{\bar{t}}$ obtained in Lemma A.16. This leads to a contradiction with the existence of a $B_\epsilon(t_0) \subset (t_S, \bar{t})$ such that the indifference condition (B.16) is satisfied. By right continuity of the belief process l_t, it implies that $l_t = \min S(t)$ for all $t \in [t_S, \bar{t})$. □

Lemma B.6 (Beliefs at t_S)**.** *Suppose belief process $(l_t)_{t \geq 0}$ is right continuous and satisfies the lowest continuation surplus refinement 1. Then $S(t_S) = \min S(\bar{t})$ and $A(\pi(m_t,t)) - A(\pi(l_t,t)) \leq rV$ for all*

$t \in [t_S, \bar{t}]$, *with the inequality being strict for some* $t \in [t_S, \bar{t})$.

Proof. Following Lemma B.5, the off-path belief during the quiet period $(t_S, \bar{t})$ is $l_t = \min S(\bar{t})$. The restriction on beliefs to be right continuous then requires that $l_{t_S} = \min\{S(\bar{t})\}$. The optimality of stopping at time t_S requires that

$$A(q_{t_S}) - A(\pi(l_{t_S}, t_S)) \le rV.$$

Since $(t_S, \bar{t}) \cap \mathbb{T} = \emptyset$ and $A(\pi(p,t))$ is weakly concave in t, it follows that

$$A(\pi(m_t, t)) - A(\pi(l_{t_S}, t)) \le rV \qquad \forall t \in [t_S, \bar{t}]. \tag{B.21}$$

Suppose that (B.21) holds with equality for all $t \in [t_S, \bar{t})$. This, however, implies a contradiction with the optimality of separation at time t_S as by waiting until $\bar{t}$ the agents can be perceived as $l_{\bar{t}} = m_{\bar{t}} > l_{t_S}$ following Lemma A.16. Consequently, (B.21) must be strict for some $t \in [t_S, \bar{t})$.

Suppose that $|S(t_S)| > 1$. It follows from Bayesian consistency that $\mathrm{E}\left[\tilde{p} \,|\, \tau = t_S, X_{t_S} = \mu t_S\right] = l_{t_S} = \min S(\bar{t})$. This implies that there exists a $p' > l_{t_S} \in S^I(t_S)$, implying that $|S^I(t_S) \cap S^I(\bar{t})| > 1$. It then follows from Lemma B.1 that $S^I(t_S) = S^I(\bar{t})$. This implies that the indifference belief, defined in (B.13) satisfies $L_{t_S}(p) = l_{t_S}$ and $L_{\bar{t}}(p) = l_{\bar{t}}$ for all $p \in S^I(t_S) = S^I(\bar{t})$. The dynamics of the indifference belief process $L_t(p)$ in (B.14) combined with the necessary condition (B.21) implies that $L_{t_S}(p)$ is strictly increasing in p for all $s \in [t_S, \bar{t})$. This leads to a contradiction with $S^I(t_S) = S^I(\bar{t})$. □

Lemma B.7 (Kink at t_S)**.** *The joint welfare function* $W_t(p)$ *has a positive kink at* $p = l_{t_S}$, *i.e.,* $W_t'(l_{t_S}-) < W_t'(l_{t_S}+)$.

Proof. The joint welfare function $W_t(p)$ is linear for $p \in [l_{t_S}, \overline{p}]$. If it were the case that $W_t'(l_{t_S}-) = W_t'(l_{t_S}+)$, then it would imply that types $t \in (l_{t_S}, \overline{p}]$ are indifferent between separating at time t_S and waiting until $\bar{t}$. This, however, contradicts the second part of Lemma B.6 that waiting between t_S and $\bar{t}$ is costly. □

Denote by t_Q the last instance prior to t_S such that there does not exist any separations prior to t_Q, i.e.,

$$t_Q \overset{def}{=} \sup\left\{t \le t_S : \quad \exists \varepsilon > 0 \quad \text{s.t.} \quad (t-\varepsilon, t) \notin \mathbb{T}^I\right\}. \tag{B.22}$$

Lemma B.8 (Separation during (t_Q, t_S))**.** *Suppose* $(l_t)_{t\ge 0}$ *is right continuous and the equilibrium satisfies the lowest continuation surplus refinement 1. Then there is no pooling for any* $t \in [t_Q, t_S]$, *i.e.,* $|S(t)| = 1$.

Proof. If $t_Q = t_S$, then the statement of Lemma B.8 follows from Lemma B.6. Suppose $t_Q < t_S$. By definition of t_Q, the set of separating times $\mathbb{T}^I$ is dense in (t_Q, t_S). The optimality of separations over the dense period (t_Q, t_S) requires that l_t is continuous for $t \in (t_Q, t_S)$. The first order optimality condition for type $p \in S(t)$ is

$$A(\pi(m_t, t)) - rV - A(\pi(l_t, t)) + \dot{l}_t \cdot \partial_1 \pi(l_t, t) \cdot \partial_2 U(\pi(p, t), \pi(l_t, t)) = 0. \tag{B.23}$$

Optimality condition (B.23) then also requires for l_t to be differentiable for all $t \in (t_Q, t_S)$. Solving (B.23) for $\dot{l}_t$ obtain

$$\dot{l}_t = \frac{A(\pi(l_t, t)) + rV - A(\pi(m_t, t))}{\partial_1 \pi(l_t, t) \cdot \partial_2 U(\pi(p, t), \pi(l_t, t))}. \tag{B.24}$$

Moreover, the linearity of $U(\pi(p, t), \pi(l, t))$ in $\pi(p, t)$ implies that process l_t satisfies

$$\dot{l}_t = \frac{A(\pi(l_t, t)) + rV - A(\pi(m_t, t))}{\partial_1 \pi(l_t, t) \cdot \partial_2 U(\pi(l_t, t), \pi(l_t, t))}. \tag{B.25}$$

The local indifference condition for the separating type $p = l_t$ requires that

$$\dot{l}_t = \frac{A(\pi(l_t, t)) + rV - A(\pi(m_t, t))}{\partial_1 \pi(l_t, t) \cdot \partial_2 U(\pi(l_t, t), \pi(l_t, t))} \qquad \forall t \in [t_Q, t_S]. \tag{B.26}$$

(i) Suppose $A(\pi(l_t, t)) + rV - A(\pi(m_t, t)) \neq 0$ for a given t. In this case, local optimality condition (B.23) cannot be satisfied by multiple types $p \in S^I(t)$ due to strict monotonicity of $\partial_2 U(\pi(p, t), \pi(l_t, t))$ in p. The strict optimality of separation times during $[t_Q, t_S]$ implies that the joint welfare $W_0(p)$ is strictly convex at $p = l_t$, implying that $l_t \notin S^I(\bar{t})$ for $t < t_S$. From here, it follows that $\dot{l}_{t_S} > 0$ since, otherwise, it would contradict the linearity of $W_0(p)$ for $p \in S^I(\bar{t})$.

(ii) Consider now the set of times

$$\hat{\mathbb{T}} \stackrel{def}{=} \{t \in [t_Q, t_S] : \quad A(\pi(l_t, t)) + rV - A(\pi(m_t, t)) = 0\}.$$

Given the continuity of l_t for $t \in [t_Q, t_S]$ it follows set $\hat{\mathbb{T}}$ is closed. It follows from (B.25) that $\dot{l}_t = 0$ for every $t \in \hat{\mathbb{T}}$. Consider two cases.

   (i) Suppose set $\hat{\mathbb{T}}$ is nowhere dense. Then for every $\hat{t} \in \hat{\mathbb{T}}$ there exist a $\hat{t}_\varepsilon \notin \hat{\mathbb{T}}$ such that $|\hat{t} - \hat{t}_\varepsilon| < \varepsilon$ for every $\varepsilon > 0$. This implies that $W_0(p)$ is strictly convex at every $p = l_{\hat{t}_\varepsilon}$, implying that there cannot be pooling at $\hat{t}_\varepsilon$.

   (ii) Suppose segment $[t_1, t_2] \subseteq [t_Q, t_S]$ is dense in $\hat{\mathbb{T}}$. By continuity of l_t over $[t_Q, t_S]$ it follows that $[t_1, t_2] \in \hat{\mathbb{T}}$. It follows from (B.25) that $\dot{l}_t = 0$ for all $t \in [t_1, t_2]$, implying that $l_t = l_{t_1}$ for every $t \in [t_1, t_2]$. This implies that $S^I(t) = S^I(t_1) = S^I(t_2)$ for any $t \in (t_1, t_2)$. Suppose that $|S^I(t_2)| > 1$ meaning that pooling is incentive compatible during the period $[t_1, t_2]$. Since Lemma B.6 proves that $|S^I(t_S)| = 1$, it follows that $t_2 < t_S$. This implies that the joint welfare function

$W_0(p)$ is strictly convex over $[l_{t_2}, l_{t_S}]$. This implies that $S^I(t_1) = S^I(t_2) < l_{t_2}$. Consequently, if there is any pooling during $[t_1, t_2]$, it would lead to a positive jump in beliefs at t_2, which contradicts continuity of l_t during $[t_Q, t_S]$. Consequently, there cannot be pooling during the period $[t_1, t_2]$.

□

Lemma B.9 (Uniqueness of smooth separation times)**.** *There does not exist a $B_\varepsilon(p_0) \subset \left[l_{t_Q}, l_{t_S}\right]$ and times $B_{\varepsilon_1}(t_1)$ and $B_{\varepsilon_2}(t_2)$, such that $B_{\varepsilon_1}(t_1) \cap B_{\varepsilon_2}(t_2) = \emptyset$, for which*

(i) the set of separation times $\cup_{p \in B_\varepsilon(p_0)} \mathbb{T}^I(p)$ is dense in $B_{\varepsilon_i}(t_i)$ for $i \in \{1, 2\}$;

(ii) each $B_{\varepsilon_i}(t_i)$ contains a weakly optimal separation time, i.e., $\mathbb{T}^I(p) \cap B_{\varepsilon_i}(t_i) \neq \emptyset$ for each $p \in B_\varepsilon(p_0)$;

(iii) the first order condition holds at each $t \in \left\{\cup_{p \in B_\varepsilon(p_0)} \mathbb{T}^I(p)\right\} \cap \{B_{\varepsilon_1}(t_1) \cup B_{\varepsilon_2}(t_2)\}$, i.e.,

$$A(\pi(m_t, t)) - rV - A(\pi(l_t, t)) + \dot{l}_t \cdot \partial_1 \pi(l_t, t) \cdot \partial_2 U(\pi(l_t, t), \pi(l_t, t)) = 0. \tag{B.27}$$

Proof. The optimal separation time solves $\tau(p)$ for an (ex-ante) type p solves

$$\begin{aligned} W_0(p) = \sup_\tau \Bigg\{ &\int_0^\tau e^{-rt} \cdot \left(p + (1-p) \cdot e^{-\lambda t}\right) \cdot \left(A(\pi(m_t, t)) - rV\right) dt \\ &+ e^{-r\tau} \cdot \left[p \cdot u_1(\pi(l_\tau, \tau)) + (1-p) e^{-\lambda \tau} \cdot u_0(\pi(l_\tau, \tau))\right] + V \Bigg\} \\ \overset{(i)}{=} &\left\{ \int_0^\tau e^{-rt} \cdot \left(p + (1-p) e^{-\lambda t}\right) \cdot \Big(A(\pi(m_t, t)) - rV - A(\pi(p, t))\Big)\, dt \right\} + U(p, p) + V, \end{aligned} \tag{B.28}$$

where equality (i) holds because, following Lemma B.8, $l_{t(p)} = p$ for all $t(p) \in [t_Q, t_S]$.

Suppose from the contrary that there exist $B_\varepsilon(p_0)$ and $B_{\varepsilon_1}(t_1)$ and $B_{\varepsilon_2}(t_2)$ satisfying the conditions (i), (ii), and (iii) of Lemma B.9. If $p \in B_\varepsilon(p_0)$, condition (ii) implies that there exist two solutions $t_1(p) < t_2(p)$ to (B.28) such that $t_1(p) \in B_{\varepsilon_1}(p)$ and $t_2(p) \in B_{\varepsilon_2}(p)$. As both stopping times are optimal, and lead to the same expected joint welfare, it implies that

$$0 = \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \left(p + (1-p) \cdot e^{-\lambda t}\right) \cdot \left[A(\pi(m_t, t)) - rV - A(\pi(p, t))\right] dt. \tag{B.29}$$

Following the argument of Lemma B.8, conditions (i) and (iii) require that $t_1(p)$ and $t_2(p)$ are continuous in p for $p \in B_\varepsilon(p_0)$. Differentiate identity (B.29) with respect to p to obtain

$$0 = \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \left(1 - e^{-\lambda t}\right) \cdot \Big[A(\pi(m_t, t)) - rV - A(\pi(p, t))\Big]\, dt$$

$$
\begin{aligned}
&- \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \Big(p + (1-p) \cdot e^{-\lambda t}\Big) \cdot A'(\pi(p,t)) \cdot \partial_1 \pi(p,t) \, dt \\
&+ e^{-rt_2(p)} \cdot \Big(p + (1-p)e^{-\lambda t_2(p)}\Big) \cdot \Big[A\Big(\pi(m_{t_2(p)}, t_2(p))\Big) - rV - A(\pi(p, t_2(p)))\Big] \cdot t_2'(p) \\
&- e^{-rt_1(p)} \cdot \Big(p + (1-p)e^{-\lambda t_1(p)}\Big) \cdot \Big[A\Big(\pi(m_{t_1(p)}, t_1(p))\Big) - rV - A(\pi(p, t_1(p)))\Big] \cdot t_1'(p) \\
&\stackrel{(i)}{=} \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \Big(1 - e^{-\lambda t}\Big) \cdot [A(\pi(m_t, t)) - rV - A(\pi(p,t))] \, dt \\
&- \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \Big(p + (1-p) \cdot e^{-\lambda t}\Big) \cdot A'(\pi(p,t)) \cdot \partial_1 \pi(p,t) \, dt \\
&- e^{-rt_2(p)} \cdot \Big(p + (1-p)e^{-\lambda t_2(p)}\Big) \cdot \partial_1 \pi(p, t_2(p)) \cdot \partial_2 U(\pi(p, t_2(p)), \pi(p, t_2(p))) \\
&+ e^{-rt_1(p)} \cdot \Big(p + (1-p)e^{-\lambda t_1(p)}\Big) \cdot \partial_1 \pi(p, t_1(p)) \cdot \partial_2 U(\pi(p, t_1(p)), \pi(p, t_1(p))),
\end{aligned}
$$

where equality (i) holds by condition (iii) of Lemma B.9. Note that

$$
\begin{aligned}
& e^{-rt} \cdot \Big(p + (1-p)e^{-\lambda t}\Big) \cdot \partial_1 \pi(p,t) \cdot \partial_2 U(\pi(p,t), \pi(p,t)) \\
=& e^{-rt} \cdot \Big(p + (1-p)e^{-\lambda t}\Big) \cdot \frac{e^{-\lambda t}}{(p + (1-p)e^{-\lambda t})^2} \\
\times & \int_0^\infty e^{-rs} \Big(\pi(p,t) + (1-\pi(p,t))e^{-\lambda s}\Big) \cdot A'(\pi(p, t+s)) \cdot \frac{e^{-\lambda s}}{\big(\pi(p,t) + (1-\pi(p,t))e^{-\lambda s}\big)^2} \, ds \\
=& \frac{e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} \cdot \int_0^\infty e^{-r(s+t)} \cdot \frac{p + (1-p)e^{-\lambda t} \cdot e^{-\lambda s}}{p + (1-p)e^{-\lambda t}} \cdot \frac{A'(\pi(p,t+s)) \cdot e^{-\lambda s}}{\Big(\frac{p}{p+(1-p)e^{-\lambda t}} + \frac{(1-p)e^{-\lambda t}}{p+(1-p)e^{-\lambda t}} e^{-\lambda s}\Big)^2} \, ds \\
=& \int_0^\infty e^{-r(s+t)} \Big(p + (1-p)e^{-\lambda(s+t)}\Big) \cdot \frac{A'(\pi(p,t+s)) \cdot e^{-\lambda(s+t)}}{(p + (1-p)e^{-\lambda(s+t)})^2} \, ds \\
=& \int_0^\infty e^{-r(s+t)} \cdot \frac{A'(\pi(p,t+s)) \cdot e^{-\lambda(s+t)}}{p + (1-p)e^{-\lambda(s+t)}} \, ds = \int_t^\infty e^{-rs} \cdot \frac{A'(\pi(p,s)) \cdot e^{-\lambda s}}{p + (1-p)e^{-\lambda s}} \, ds.
\end{aligned}
$$

The above equality implies that

$$
\begin{aligned}
0 =& \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \Big(1 - e^{-\lambda t}\Big) \cdot [A(\pi(m_t, t)) - rV - A(\pi(p,t))] \, dt \\
&- \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \Big(p + (1-p) \cdot e^{-\lambda t}\Big) \cdot A'(\pi(p,t)) \cdot \partial_1 \pi(p,t) \, dt \\
&- e^{-rt_2(p)} \cdot \Big(p + (1-p)e^{-\lambda t_2(p)}\Big) \cdot \partial_1 \pi(p, t_2(p)) \cdot \partial_2 U(\pi(p, t_2(p)), \pi(p, t_2(p))) \\
&+ e^{-rt_1(p)} \cdot \Big(p + (1-p)e^{-\lambda t_1(p)}\Big) \cdot \partial_1 \pi(p, t_1(p)) \cdot \partial_2 U(\pi(p, t_1(p)), \pi(p, t_1(p)))
\end{aligned}
$$

$$
\begin{aligned}
0 = & \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \left(1 - e^{-\lambda t}\right) \cdot \left[A(\pi(m_t, t)) - rV - A(\pi(p, t))\right] dt \\
& - \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot A'(\pi(p, t)) \cdot \frac{e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} \, dt \\
& - \int_{t_2(p)}^{+\infty} e^{-rt} \cdot A'(\pi(p, t)) \cdot \frac{e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} \, dt + \int_{t_1(p)}^{+\infty} e^{-rt} \cdot A'(\pi(p, t)) \cdot \frac{e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} \, dt \\
0 = & \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \left(1 - e^{-\lambda t}\right) \cdot \Big[A(\pi(m_t, t)) - rV - A(\pi(p, t))\Big] \, dt. \qquad \text{(B.30)}
\end{aligned}
$$

Combine (B.29) and (B.30) to obtain

$$
\begin{cases}
\displaystyle \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot \left[A(\pi(m_t, t)) - rV - A(\pi(p, t))\right] dt = 0, \\
\displaystyle \int_{t_1(p)}^{t_2(p)} e^{-(r+\lambda)t} \cdot \left[A(\pi(m_t, t)) - rV - A(\pi(p, t))\right] dt = 0.
\end{cases} \qquad \text{(B.31)}
$$

which holds for all $p \in B_\varepsilon(p_0)$. Differentiate the top equality in (B.31) with respect to p to obtain

$$
\begin{aligned}
0 = & \, e^{-rt_2(p)} \cdot \Big[A\Big(\pi(m_{t_2(p)}, t_2(p))\Big) - rV - A(\pi(p, t_2(p)))\Big] \cdot t_2'(p) \\
& - e^{-rt_1(p)} \cdot \Big[A\Big(\pi(m_{t_1(p)}, t_1(p))\Big) - rV - A(\pi(p, t_1(p)))\Big] \cdot t_1'(p) \\
& - \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot A'(\pi(p, t)) \cdot \partial_1 \pi(p, t) \, dt \\
0 = & -e^{-rt_2(p)} \cdot \partial_1 \pi(p, t_2(p)) \cdot \partial_2 U(\pi(p, t_2(p)), \pi(p, t_2(p))) \\
& + e^{-rt_1(p)} \cdot \partial_1 \pi(p, t_1(p)) \cdot \partial_2 U(\pi(p, t_1(p)), \pi(p, t_1(p))) \\
& - \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot A'(\pi(p, t)) \cdot \frac{e^{-\lambda t}}{(p + (1-p)e^{-\lambda t})^2} \, dt \\
0 = & -\frac{1}{p + (1-p)e^{-\lambda t_2(p)}} \cdot \int_{t_2(p)}^{\infty} e^{-rt} \cdot \frac{A'(\pi(p, t)) \cdot e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} \, dt \\
& + \frac{1}{p + (1-p)e^{-\lambda t_1(p)}} \cdot \int_{t_1(p)}^{\infty} e^{-rt} \cdot \frac{A'(\pi(p, t)) \cdot e^{-\lambda t}}{p + (1-p)e^{-\lambda t}} \, dt \\
& - \int_{t_1(p)}^{t_2(p)} e^{-rt} \cdot A'(\pi(p, t)) \cdot \frac{e^{-\lambda t}}{(p + (1-p)e^{-\lambda t})^2} \, dt \\
0 = & \int_{t_1(p)}^{t_2(p)} e^{-(r+\lambda)t} \cdot \frac{A'(\pi(p, t))}{p + (1-p)e^{-\lambda t}} \cdot \underbrace{\left(\frac{1}{p + (1-p)e^{-\lambda t_1(p)}} - \frac{1}{p + (1-p)e^{-\lambda t}}\right)}_{<0} dt
\end{aligned}
$$

$$+\underbrace{\left(\frac{1}{p+(1-p)e^{-\lambda t_1(p)}}-\frac{1}{p+(1-p)e^{-\lambda t_2(p)}}\right)}_{<0}\cdot\int_{t_2(p)}^{\infty}e^{-(r+\lambda)t}\cdot\frac{A'(\pi(p,t))}{p+(1-p)e^{-\lambda t}}\,dt.$$

The above inequality cannot be satisfied since both terms on the right hand side are negative. This leads to a contradiction with the existence of $t_1(p)$ and $t_2(p)$ which are solutions (B.28), all the while the first order optimal stopping conditions (iii) in the Lemma hold.

□

Lemma B.10 (Increasing separations between t_Q and t_S)**.** *Gradual separations during* $[t_Q, t_S]$ *are increasing in types, as captured by* $\dot{l}_t \geq 0$ *for all* $t \in (t_Q, t_S)$.

Proof. Following Lemma B.8, process l_t is differentiable for $t \in [t_Q, t_S]$. Suppose that l_t is non-monotone, meaning that there exists a local minimum or maximum. Without loss, suppose l_t has a strict local minimum at $t_0 \in (t_Q, t_S)$. This implies that there exists a pair $t_1 < t_0 < t_2$ such that

$$l_{t_1} > l_{t_0} > l_{t_2}.$$

Consider the type

$$\hat{p} = \frac{l_{t_0} + \min\{l_{t_1}, l_{t_2}\}}{2}.$$

Then there exists a $\hat{t}_1 < t_0 < \hat{t}_2$ such that $l_{\hat{t}_1} = l_{\hat{t}_2} = \hat{p}$. This, however, implies a contradiction with Lemma B.9 as there exists a neighborhood $B_\varepsilon(\hat{p})$ that find it optimal to separate both during $B_{\epsilon_1}(\hat{t}_1)$ and $B_{\epsilon_2}(\hat{t}_2)$ with $B_{\epsilon_1}(\hat{t}_1) \cap B_{\epsilon_2}(\hat{t}_2) = \emptyset$ and the first order optimality condition (B.27) holding due to Lemma B.8. This is a contradiction with t_0 being a local maximum of l_t during $[t_Q, t_S]$.

Suppose belief process l_t is decreasing during $[t_Q, t_S]$, implying that $l_{t_Q} > l_{t_S}$. Due to the first order separating condition (B.23) is stricly increasing in type p, it implies that the joint welfare function $W_0(p)$ is strictly convex for $p \in [l_{t_Q}, l_{t_S}]$. Following Lemma B.6, however, this welfare function $W_0(p)$ is linear for $p \in [l_{t_S}, \bar{p}]$, due to the indifference of type l_{t_S} in waiting until time $\bar{t}$ to separate. If $l_{t_Q} > l_{t_S}$, then $[l_{t_Q}, l_{t_S}] \subseteq [l_{t_S}, \bar{p}]$, which leads to a contradiction with l_t being decreasing over $[t_Q, t_S]$. □

Define t^* to be the last time before t_Q when a type is willing to separate

$$t^* \overset{def}{=} \sup\left\{t < t_Q : \quad t \in \mathbb{T}^I\right\} \in [0, t_Q). \tag{B.32}$$

By definition of t_Q, it follows that $t^* < t_Q$ if such a t^* exists.

Lemma B.11 (Quiet period beliefs)**.** *Suppose the equilibrium satisfies the lowest continuation surplus refinement 1 and belief process* $l = (l_t)_{t \geq 0}$ *is right continuous. If there exists* $t^* \geq 0$ *then* $l_{t^*} \in (l_{t_Q}, l_{t_S}]$

and process $l = (l_t)_{t\geq 0}$ *satisfies*

$$\int_t^{t(l_t)} \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds = 0 \tag{B.33}$$

and

$$\text{sign}\left(\dot{l}_t\right) = \text{sign}\left\{\int_t^{t(l_t)} e^{-r(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds\right\}. \tag{B.34}$$

for all $t \in (t^*, t^* + \varepsilon)$ *for some* $\varepsilon > 0$.

Proof. The joint continuation welfare (B.2) is given by

$$W_t(p) = \sup_\tau \Bigg\{\int_t^\tau e^{-r(s-t)} \cdot \left(\pi(p,t) + (1-\pi(p,t)) \cdot e^{-\lambda(s-t)}\right) \cdot (A(\pi(m_s, s)) - rV)\; ds$$
$$+ e^{-r(\tau-t)} \cdot \Big[\pi(p,t) \cdot u_1(\pi(l_\tau, \tau)) + (1-\pi(p,t)) \cdot e^{-\lambda(\tau-t)} \cdot u_0(\pi(l_\tau, \tau))\Big]\Bigg\} + V.$$

Following Lemma B.8, this joint welfare $W_t(p)$ is strictly convex in $\pi(p,t)$ for $p \in (l_{t_Q}, l_{t_S})$ as each type has a uniquely optimal date over when to separate. The off-path belief l_t solves

$$V_t(l_t) = W_t(l_t) - U_t(l_t) = \min_{x\in R(t)} V_t(x) = \min_{x\in R(t)} (W_t(x) - U_t(x)) = \min_{x\in[l_{t_Q}, \bar{p}]} (W_t(x) - U_t(x)). \tag{B.35}$$

Suppose that $l_{t^*} = l_{t_Q}$**.** Following Lemmas B.5, A.16, and B.10 it implies that the set of remaining types $R(t^*) \subseteq [l_{t_Q}, \bar{p}]$. From the lowest continuation surplus refinement, it implies that $l_t \geq l_{t_Q}$ for $t \geq t^*$. In order for type $p = l_{t_Q}$ to find it incentive compatible to separate at time t^* rather than at time $t^* + \epsilon$ it must be the case that

$$A(\pi(m_{t^*}, t^*) - rV - A(\pi(l_{t^*}, t^*)) \leq 0. \tag{B.36}$$

For this same type $p = l_{t_Q}$ to be willing to wait until time t_Q, it then is necessary that (B.36) is binding for all $t \in [t^*, t_Q]$ due to concavity of $A(\pi(\underline{p}, t))$ in t. This, however, contradicts the definition of (B.32) as it means tyat type $p = l_{t_Q}$ finds it weakly optimal to separate anytime during $[t^*, t_Q]$. This implies a contradiction with $l_{t^*} = l_{t_Q}$.

Suppose that $l_{t^*} > l_{t_S}$**.** By right continuity there exists an $\varepsilon > 0$ such that $l_t > l_{t_S}$ for every $t \in [t^*, t^* + \varepsilon)$. This implies that $V_t'(p) = 0$ for all $p \in (l_{t_S}, \bar{p}]$ and $t \in [t^*, t^* + \varepsilon)$. This, however, contradicts the argument of Lemma B.5 which showed that there does not existn $l_t \in (l_{t_S}, \bar{p}]$ such that $V_t'(l_t) = 0$ for t in a positive interval.

Suppose $l_{t^*} = l_{t_S}$. Suppose that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n > t^*$, $l_{t_n} \to l_{t^*}$, and $l_{t_n} \geq l_{t_S}$. In order for $t^* = t_S \in \mathbb{T}^I$ it is then necessary that

$$A(\pi(m_{t^*}, t^*)) - rV - A(\pi(l_{t^*}, t^*)) \leq 0 \tag{B.37}$$

since, otherwise, at type t_S would find is strictly preferable to separate at time t_n for n high enough. If, however, (B.37) is satisfied, then type l_{t_Q} is strictly better of separating at time t^* and be perceived as $l_{t_S} > l_{t_Q}$, rather than wait until t_Q as can be seen from

$$\begin{aligned}
W_{t^*}(l_{t_Q}) &= \int_{t^*}^{t_Q} \left(\pi(l_{t_Q}, t^*) \cdot e^{-r(t-t^*)} + (1 - \pi(l_{t_Q}, t^*)) \cdot e^{-(r+\lambda)(t-t^*)}\right) \cdot \left(A(\pi(m_t, t)) - rV\right) dt \\
&\quad + \int_{t_Q}^{\infty} \left(\pi(l_{t_Q}, t^*) \cdot e^{-r(t-t^*)} + (1 - \pi(l_{t_Q}, t^*)) \cdot e^{-(r+\lambda)(t-t^*)}\right) \cdot A(\pi(l_{t_Q}, t))\, dt + V \\
&\overset{(i)}{=} \int_{t^*}^{t_Q} \left(\pi(l_{t_Q}, t^*) e^{-r(t-t^*)} + (1 - \pi(l_{t_Q}, t^*)) e^{-(r+\lambda)(t-t^*)}\right) \underbrace{\left(A(\pi(m_{t^*}, t)) - rV - A(\pi(l_{t^*}, t))\right)}_{\leq 0} dt \\
&\quad + \int_{t_Q}^{\infty} \left(\pi(l_{t_Q}, t^*) \cdot e^{-r(t-t^*)} + (1 - \pi(l_{t_Q}, t^*)) \cdot e^{-(r+\lambda)(t-t^*)}\right) \underbrace{\left(A(\pi(l_{t_Q}, t)) - A(\pi(l_{t^*}, t))\right)}_{<0} dt \\
&\quad + \int_{t^*}^{\infty} \left(\pi(l_{t_Q}, t^*) \cdot e^{-r(t-t^*)} + (1 - \pi(l_{t_Q}, t^*)) \cdot e^{-(r+\lambda)(t-t^*)}\right) \cdot A(\pi(l_{t^*}, t))\, dt + V \\
&\overset{(ii)}{<} \int_{t^*}^{\infty} \left(\pi(l_{t_Q}, t^*) \cdot e^{-r(t-t^*)} + (1 - \pi(l_{t_Q}, t^*)) \cdot e^{-(r+\lambda)(t-t^*)}\right) \cdot A(\pi(l_{t^*}, t))\, dt + V.
\end{aligned}$$

Equality (i) follows from the fact that $(t^*, t_Q) \not\subset \mathbb{T}$ and hence $m_t = m_{t^*}$ during that period. Strict inequality (ii) follows from weak inequality (B.37), combined with weak concavity $A(\pi(\underline{p}, t))$ in t and strict inequality $l_{t_S} > l_{t_Q}$ which follows from Lemma B.10.

Remaining cases. The above arguments imply that $l_{t^*} \leq l_{t_S}$ and $l_t \in (l_{t_Q}, l_{t_S})$ for all $t \in [t^*, t^* + \varepsilon)$ for some $\epsilon > 0$. This implies that for each $t \in (t^*, t^* + \varepsilon)$ the off-equilibrium path belief l_t must be an interior solution to (B.35) due to strict convexity of $W_t(p)$ for $p \in (l_{t_Q}, l_{t_S})$ as derived in Lemma B.8. Such an interior condition must satisfy

$$W_t'(l_t) = U_t(l_t) \qquad \Leftrightarrow \qquad W_t'(l_t) = [u_1(\pi(l_t, t)) - u_0(\pi(l_t, t))] \cdot \partial_1 \pi(l_t, t).$$

Use the Envelope theorem to compute the derivative of $W_t(p)$

$$\begin{aligned}
\frac{W_t'(p)}{\partial_1 \pi(p, t)} = &\int_t^{t(p)} e^{-r(s-t)} \left(1 - e^{-(r+\lambda)(s-t)}\right) \cdot (A(\pi(m_s, s)) - rV)\; ds \\
&+ e^{-r(t(p)-t)} \cdot u_1(\pi(l_{t(p)}, t(p))) - e^{-(r+\lambda)(t(p)-t)} \cdot u_0(\pi(l_{t(p)}, t(p))).
\end{aligned}$$

This implies that l_t satisfies $W_t'(p) = U_t'(p)$ if

$$\int_t^{t(l_t)} \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds = 0, \tag{B.38}$$

which yields (B.33). The right continuity of the belief process l_t implies that (B.33) must hold in some neighborhood $B_\varepsilon(t_0) \not\subset \mathbb{T}$. The derivative of (B.33) with respect to t is

$$\begin{aligned} 0 =& \left(e^{-r(t(l_t)-t)} - e^{-(r+\lambda)(t(l_t)-t)}\right) \cdot \overbrace{\Big(A(\pi(m_{t(l_t)}, t(l_t))) - rV - A(\pi(l_t, T(l_t)))\Big) \cdot t'(l_t)}^{\leq 0} \cdot \dot{l}_t \\ & \underbrace{- \int_t^{t(l_t)} \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot A'(\pi(l_t, s)) \cdot \partial_1 \pi(l_t, s)\, ds}_{\geq 0} \cdot \dot{l}_t \\ & + \int_t^{t(l_t)} \left(-re^{-r(s-t)} + (r+\lambda)e^{-(r+\lambda)(s-t)}\right) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds. \end{aligned} \tag{B.39}$$

Use the first order condition $t'(l_t)$ and (B.33), to simplify (B.39) to

$$\begin{aligned} 0 =& - \left(e^{-r(t(l_t)-t)} - e^{-(r+\lambda)(t(l_t)-t)}\right) \cdot \partial_2 U(\pi(l_t, t(l_t)), \pi(l_t, t(l_t)) \cdot \partial_1 \pi(l_t, t(l_t)) \cdot \dot{l}_t \\ & - \int_t^{t(l_t)} \left(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\right) \cdot A'(\pi(l_t, s)) \cdot \partial_1 \pi(l_t, s)\, ds \cdot \dot{l}_t \\ & + \int_t^{t(l_t)} \lambda e^{-(r+\lambda)(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds. \end{aligned} \tag{B.40}$$

The terms multiplying $\dot{l}_t$ in (B.40) have the same sign (negative), implying that

$$\begin{aligned} \operatorname{sign}\left(\dot{l}_t\right) &= \operatorname{sign}\left\{ \int_t^{t(l_t)} e^{-(r+\lambda)(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds \right\} \\ &\overset{(i)}{=} \operatorname{sign}\left\{ \int_t^{t(l_t)} e^{-r(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds \right\}, \end{aligned}$$

where (i) holds due to (B.33). This yields (B.34). □

Lemma B.12 (Single separating period)**.** *Suppose belief process $(l_t)_{t\geq 0}$ is right continuous and satisfies the lowest continuation surplus refinement 1. Then there does not exist a t^* satisfying definition* (B.32).

Proof. Following Lemma B.11, it follows that $l_{t^*} \in (l_{t_Q}, l_{t_S}]$. Moreover, the requirement of the lowest continuation surplus refinement 1 that $l_t \in R(t)$ implies that $l_t \geq l_{t_Q}$ for all $t \in (t^*, t_Q)$ due to Lemma B.10. For type l_{t_Q} to find it incentive compatible to wait until time t_Q to separate, it must be the case that $A(\pi(m_t, t)) - A(\pi(l_{t_Q}, t)) - rV \geq 0$ in the vicinity of t_Q. Moreover, weak concavity of $A(\pi(p, t))$ in

t and the fact that there are no separations during $[t^*, t_Q]$ by definition of t^* implies that

$$A(\pi(m_t, t)) - A(\pi(l_{t_Q}, t)) \geq rV \qquad \textit{for all} \qquad t \in [t^*, t_Q]. \tag{B.41}$$

(i) Suppose $A\left(\pi(m_{t^*}, t^*)\right) - A\left(\pi(l_{t_Q}, t^*)\right) = rV$. From weak concavity of $A(\pi(p,t))$ in t, it implies that $A(\pi(m_t, t)) - A(\pi(l_{t_Q}, t)) - rV = 0$ for all $t \in [t^*, t_Q]$. This implies that $l_{t^*} = l_{t_Q}$ and, moreover, that this type is indifferent in separating anytime between t^* and t_Q, contradicting the definition of t_Q.

(ii) Suppose $A\left(\pi(m_{t^*}, t^*)\right) - A\left(\pi(l_{t_Q}, t^*)\right) > rV$. Following Lemma B.11, it must be the case that $l_{t^*} \in (l_{t_Q}, t_S]$. By right continuity, following Lemma B.11 it implies that

$$\int_{t^*}^{t(l_{t^*})} \left(e^{-r(s-t^*)} - e^{-(r+\lambda)(s-t^*)}\right) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds = 0. \tag{B.42}$$

(a) First, consider the case

$$\begin{aligned} 0 &< \int_{t^*}^{t(l_{t^*})} e^{-r(s-t^*)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t'_S}, s))\Big)\, ds \\ &\overset{(i)}{=} \int_{t^*}^{t(l_{t^*})} e^{-(r+\lambda)(s-t^*)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds, \end{aligned}$$

where equality (i) follows from (B.42). This implies that

$$\int_{t^*}^{t(l_{t^*})} \left(l_{t^*} \cdot e^{-r(s-t^*)} + (1 - l_{t^*}) \cdot e^{-(r+\lambda)(s-t^*)}\right) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds > 0,$$

which implies that type l_{t^*} finds it strictly optimal to wait until time $t(l_{t^*}) \in (t_Q, t_S)$ to separate, rather than separate at time t^*. This is a contradiction.

(b) Second, consider the case

$$\begin{aligned} 0 &> \int_{t^*}^{t(l_{t^*})} e^{-r(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds \\ &\overset{(i)}{=} \int_{t'_S}^{t(l_{t^*})} e^{-(r+\lambda)(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds, \end{aligned}$$

where equality (i) follows from (B.42). Similar to the previous case, type l_{t^*} then finds it strictly optimal to separate at time t^*, rather than wait until $t(l_{t^*}) \in [t_Q, t_S]$, which is a contradiction.

(c) Third, and final, consider the case

$$0 = \int_{t^*}^{t(l_{t^*})} e^{-r(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds. \tag{B.43}$$

Following (B.34) of Lemma B.11, it implies that $\dot{l}_{t^*} = 0$. This implies that for separation to be optimal for type l_{t^*} at time t^*, it requires that $A(q_{t^*}) - A(\pi(l_{t^*}, t^*)) < rV$. Rewrite (B.43) as

$$0 = \overbrace{\int_{t^*}^{t_Q} e^{-r(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds}^{\overset{(i)}{<}0} + \underbrace{\int_{t_Q}^{t(l_{t^*})} e^{-r(s-t)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t^*}, s))\Big)\, ds}_{\overset{(ii)}{<}0} < 0, \tag{B.44}$$

Inequality (i) in (B.44) for the first term follows from $A(q_{t^*}) - A(\pi(l_{t^*}, t^*)) < rV$ and weak concavity of $A(\pi(\underline{p}, t))$ in t. Inequality (ii) in (B.44) for the second term follows from Lemma B.10 and the fact that $l_{t^*} \geq l_t$ for $t < t(l_{t^*})$. Inequality (B.44) is then a contradiction with (B.43), implying that such a case is also inconsistent with equilibrium.

□

Lemma B.13 now concludes the proof of Proposition B.1 by showing that client beliefs track the lowest ex-ante type during the quiet period.

Lemma B.13 (Unique continuous equilibrium)**.** *Suppose belief process $(l_t)_{t\geq 0}$ is continuous and satisfies the lowest continuation surplus refinement 1. Then $t_S = \bar{t}$ and $l_t = \underline{p}$ for $t \leq t_Q$.*

Proof. Suppose $t_S < \bar{t}$. It is necessarily the case that $|S(\bar{t})| > 1$ since, otherwise, waiting between t_S and $\bar{t}$ would not be incentive compatible for type l_{t_S}. Following Lemma B.5 it implies that $l_t = \min S(\bar{t})$ for every $t \in [t_S, \bar{t})$. Consequently, it implies that belief process l_t would have to have a jump at $\bar{t}$ which is a contradiction with continuity. This implies that it must be the case that $t_S = \bar{t}$.

Now, consider the off-equilibrium path beliefs for $t \leq t_Q$. The claim is that if l_t is continuous, then it must be the case that $l_t = \underline{p}$. Consider $t < t_Q$ and suppose that l_t is an interior solution to (B.15), rewritten here as

$$V_t(l_t) = \min_{p \in [\underline{p}, \overline{p}]} V_t(p). \tag{B.45}$$

Since $V_t(p) = W_t(p) - U_t(p)$ due to strict convexity of $W_t(p)$ for $p \in [\underline{p}, \overline{p}]$ and the linearity of $U_t(p)$ in p it follows that (B.45) is satisfied if and only if $W_t'(l_t) = U_t'(l_t)$. Following the proof of Lemma B.11, and equation (B.38) specifically, obtain that $W_t'(l_t) = U_t'(l_t)$ if and only if

$$\int_t^{t(l_t)} \Big(e^{-r(s-t)} - e^{-(r+\lambda)(s-t)}\Big) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_t, s))\Big)\, ds = 0. \tag{B.46}$$

Note, that $l_t \to \underline{p}$ as $t \to t_Q$. This implies that there exists a $t_0 < t_Q$ such that $\dot{l}_{t_0} < 0$ as, otherwise, it would have to be that $l_t \equiv \underline{p}$ for all $t \in [0, t_Q]$. From (B.34), it implies that $\dot{l}_{t_0} < 0$ if and only if

$$\int_{t_0}^{t(l_{t_0})} e^{-r(s-t_0)} \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t_0}, s))\Big)\, ds < 0. \tag{B.47}$$

Inequality (B.47) combined with (B.46) then implies that

$$\int_{t_0}^{t(l_{t_0})} \Big(l_{t_0} \cdot e^{-r(s-t)} + (1 - l_{t_0}) \cdot e^{-(r+\lambda)(s-t)}\Big) \cdot \Big(A(\pi(m_s, s)) - rV - A(\pi(l_{t_0}, s))\Big)\, ds > 0,$$

implying that type l_{t_0} would find it strictly optimal to separate at time t_0, rather than wait until time $t(l_{t_0})$. Consequently, there cannot be a t_0 such that $l_{t_0} > \underline{p}$ is an interior solution to (B.15) and $\dot{l}_{t_0} < 0$. This implies that $l_t \equiv \underline{p}$ for all $t \in [0, t_Q]$. □

B.4 Perturbation Approach and Limiting Beliefs

Now, we consider a perturbation of the model in which each intermediary-agent pair receive a private relationship-specific disutility shock $\tilde{s}$ distributed according $\tilde{s} \sim \text{Exp}(\Delta)$. We show that as $\Delta \to \infty$ the equilibrium beliefs must converge to the lowest continuation surplus beliefs in definition 1. We assume each intermediary-agent pair receives a shock in period t with intensity $\varepsilon > 0$. Similar to Acemoglu and Pischke (1998), we assume these shocks are discrete – once a shock $\tilde{s}$ arrives at time t, the intermediary and the agent can separate and avoid it, or continue employment, but suffer a joint disutility equal to $-\tilde{s}$.[6] Shocks $\tilde{s}$ imply that there is always a positive probability that an agent will separate from the intermediary, implying that all beliefs can be determined by Bayes rule. Intermediary-agent pairs that have a high continuation value relative to their outside options will be more resilient to relationship-specific shocks and will, thus, be less likely to separate. The exponential distribution of the shocks corresponds to the Logit specification of a Quantal Response Equilibrium introduced in McKelvey and Palfrey (1995) and McKelvey and Palfrey (1998) and can be substantially relaxed.

The magnitude of Δ is inversely related to the magnitude and dispersion of the shock and it converges to 0 as $\Delta \to +\infty$. In what follows, we index the equilibrium parameters and values by the magnitude of the shock Δ. In addition, we refer to $e_t(p; \Delta)$ as the probability of letting go of the agent of ability p at time t following the history of good performance. In addition, we denote by $f_t(p; \Delta)$ the conditional distribution of ex-ante types p that continue to be employed by the intermediary at time t following the path of good performance.

[6]Due to the wages acting as a transferable utility, it does not matter whether it is the intermediary, the agent, or a combination of the two that receives the disutility shock.

Definition 2 (Perturbed limiting equilibrium)**.** *A limiting equilibrium is a sequence of value functions $V_t(p;\Delta_n)$, probability density processes $f_t(p;\Delta_n)$, and beliefs $l_t(\Delta_n)$ and $m_t(\Delta_n)$ such that*

(i) ***Equilibrium limit:*** *there exists a uniform limit in time t and belief p:*

$$\lim_{n\to\infty}\Big\{V_t(p;\Delta_n),\dot V_t(p;\Delta_n),f_t(p;\Delta_n),m_t(\Delta_n),l_t(\Delta_n),\gamma_t(\Delta_n)\Big\}=\Big\{V_t(p),\dot V_t(p),f_t(p),m_t,l_t,\gamma_t\Big\} \quad \text{(B.48)}$$

(ii) ***Belief consistency:*** *the limiting beliefs $(l_t,m_t,\gamma_t)_{t\geq 0}$, the implied churning strategies $(e_t(p))_{t\geq 0}$, and value functions $(V_t(p))_{t\geq 0}$ for each ex-ante private type p are an equilibrium of a game in which $\varepsilon=0$ and $\Delta=+\infty$.*

(iii) ***Belief regularity:*** *for every $t<\bar t$ the support of remaining types support$(\,p\,|\tau>t,X_t=t)$ = support(f_t) is equal to its derived set, i.e., equal to its limit points.*

Consider the limiting equilibrium $(l_t,m_t,V_t(\cdot),e_t(\cdot))_{t\geq 0}$. For this limiting equilibrium define by $R(t)$ the set of types that remain with the intermediary with positive probability by time t in the limiting equilibrium:

$$R(t)\overset{def}{=}\textit{support}(f_t)=cl\big\{p:\quad f_t(p)>0\big\}. \quad \text{(B.49)}$$

Lemma B.14 (Limiting equilibrium beliefs)**.** *The limiting equilibrium belief process satisfies the lowest continuation surplus refinement 1, i.e., $l_t\in R(t)$ and $V_t(l_t)=\min_{p\in R(t)}V_t(p)$.*

Proof. Consider a time $t\notin\mathbb{T}^I$, implying that $V_t(p)>V$ for all $p\in R(t)$. Since $R(t)$ is a closed set, it implies that $\inf_{p\in R(t)}V_t(p)>V$. The value functions $V_t(p;\Delta_n)$ and $V_t(p)$ are weakly convex in p as they are solutions to their respective optimal stopping problems. Point-wise convergence of convex functions on the interval $[0,1]$ is uniform. Consequently, there exists an N sufficiently large such that $\inf_{p\in R(t)}V_t(p;\Delta_n)>V$ for all $n>N$. For $t\notin\mathbb{T}^I$ it follows that Belief $l_t(\Delta_n)$ is given by

$$l_t(\Delta_n)=\frac{\int p\cdot f_t(p;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(p;\Delta_n)-V]}\,dp}{\int f_t(p;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(p;\Delta_n)-V]}\,dp}.$$

By definition of $R(t)$ it follows that

$$\lim_{n\to\infty}l_t(\Delta_n)=\frac{\int p\cdot f_t(p;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(p;\Delta_n)-V]}\,dp}{\int f_t(p;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(p;\Delta_n)-V]}\,dp}\overset{(i)}{\in}\left[\min R(t),\,\max R(t)\right].$$

Consider now the limit $V_t(l_t)$:

$$V_t(l_t)=V\left(\lim_{n\to\infty}l_t(\Delta_n)\right)=\lim_{n\to\infty}V_t(l_t(\Delta_n))=\lim_{n\to\infty}V_t\left(\frac{\int p\cdot f_t(p;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(p;\Delta_n)-V]}\,dp}{\int f_t(p;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(p;\Delta_n)-V]}\,dp}\right)$$

$$
\begin{aligned}
&\overset{(i)}{\leq} \lim_{n\to\infty}\left[\int V_t(x)\cdot \frac{f_t(x;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(x;\Delta_n)-V]}}{\int f_t(y;\Delta_n)\cdot e^{-\Delta_n\cdot[V_t(y;\Delta_n)-V]}\,dy}\,dx\right]\\
&= \lim_{n\to\infty}\left[\int V_t(x)\cdot \frac{1}{\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]
\end{aligned}
\tag{B.50}
$$

where inequality (i) holds by Jensen's inequality due to the convexity of $V_t(p)$ in p.

Choose a $z_t \in \arg\min_{p\in R(t)} V_t(p)$. There exist $\underline{z}_{\varepsilon,t} \leq z \leq \overline{z}_{\varepsilon,t}$ with at least one of the inequalities being strict such that

(i) $(\underline{z}_{\varepsilon,t}, \overline{z}_{\varepsilon,t}) \in R(t)$, which follows from the regularity assumption that the support of $f_t(p)$ is equal to its derived set;

(ii) $V_t(z;\Delta_n) - V_t(z_t;\Delta_n) \leq \varepsilon$ for all $z \in (\underline{z}_{\varepsilon,t}, \overline{z}_{\varepsilon,t})$ and $n \geq N$, which follows from continuity and convexity of $V_t(p)$ over its domain, and uniform convergence of $V_t(p;\Delta_n)$ in p.

Define

$$
Q_\varepsilon(t) \overset{def}{=} \{x \in R(t): \quad V_t(x) > V_t(z_t) + 3\cdot\varepsilon\}\,. \tag{B.51}
$$

By uniform convergence of $V_t(x;\Delta_n)$ to $V_t(x)$ there exists N such that for all $n > N$:

$$
V_t(x;\Delta_n) > V_t(z_t;\Delta_n) + 2\cdot\varepsilon \qquad \textit{for all} \quad x \in Q_\varepsilon(t). \tag{B.52}
$$

Suppose $x \in Q_\varepsilon(t)$. Then

$$
\begin{aligned}
&\lim_{n\to\infty}\left[\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy\right]\\
&\geq \lim_{n\to\infty}\left[\int_{\underline{z}_{\varepsilon,t}}^{\overline{z}_{\varepsilon,t}} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy\right]\\
&= \lim_{n\to\infty}\left[\int_{\underline{z}_{\varepsilon,t}}^{\overline{z}_{\varepsilon,t}} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(z_t;\Delta_n)+V_t(z_t;\Delta_n)-V_t(y;\Delta_n))}\,dy\right]\\
&\geq \lim_{n\to\infty}\left[\int_{\underline{z}_{\varepsilon,t}}^{\overline{z}_{\varepsilon,t}} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(|V_t(x;\Delta_n)-V_t(z_t;\Delta_n)|-|V_t(z_t;\Delta_n)-V_t(y;\Delta_n)|)}\,dy\right]\\
&\overset{(i)}{\geq} \lim_{n\to\infty}\left[\int_{\underline{z}_{\varepsilon,t}}^{\overline{z}_{\varepsilon,t}} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(2\cdot\varepsilon-\varepsilon)}\,dy\right] = \lim_{n\to\infty}\left[e^{\Delta_n\cdot\varepsilon}\cdot\int_{\underline{z}_{\varepsilon,t}}^{\overline{z}_{\varepsilon,t}} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\,dy\right] = +\infty,\\
\Rightarrow \quad &\lim_{n\to\infty}\left[\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy\right] = +\infty \qquad \forall\, x \in Q_\varepsilon(t).
\end{aligned}
\tag{B.53}
$$

where inequality (i) follows from $y \in (\underline{z}_{\varepsilon,t}, \overline{z}_{\varepsilon,t})$ and $x \in Q_\varepsilon(t)$, which implies (B.52). Use (B.53) to simplify (B.50) as

$$\begin{aligned}
V(l_t) &\le \lim_{n\to\infty}\left[\int V_t(x)\cdot \frac{1}{\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\
&= \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} V_t(x)\cdot \frac{1}{\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\
&+ \lim_{n\to\infty}\left[\int_{Q_\varepsilon(t)} V_t(x)\cdot \frac{1}{\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\
&\overset{(i)}{=} \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} x\cdot \frac{1}{\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\
\Rightarrow\qquad V(l_t) &\le \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} x\cdot \frac{1}{\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right].
\end{aligned}\tag{B.54}$$

where equality (i) in (B.54) follows from equality (B.53).

Now, consider the denominator in (B.54). It needs to be evaluated only for $x \in R(t)\setminus Q_\varepsilon(t)$. Consider an arbitrary $\hat{\varepsilon} > \varepsilon$. By definition of $Q_\varepsilon(t)$ in (B.51) it follows that

$$V_t(y) > V_t(z_t) + 3\cdot\hat{\varepsilon} \qquad \forall y \in Q_{\hat{\varepsilon}}(t).$$

Then

$$\begin{aligned}
&\lim_{n\to\infty}\left[\int_{Q_{\hat{\varepsilon}}(t)} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy\right]\\
&= \lim_{n\to\infty}\left[\int_{Q_{\hat{\varepsilon}}(t)} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(x)+V_t(x)-V_t(y)+V_t(y)-V_t(y;\Delta_n))}\,dy\right]\\
&\le \lim_{n\to\infty}\left[\int_{Q_{\hat{\varepsilon}}(t)} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(2\cdot\max_x\{V_t(x;\Delta_n)-V_t(x)\}+V_t(x)-V_t(y))\,dy}\,dy\right]\\
&\le \lim_{n\to\infty}\left[\int_{Q_{\hat{\varepsilon}}(t)} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(2\cdot\max_x\{V_t(x;\Delta_n)-V_t(x)\}+|V_t(x)-V_t(z)|-|V_t(y)-V_t(z)|)\,dy}\,dy\right]\\
&= \lim_{n\to\infty}\left[\int_{Q_{\hat{\varepsilon}}(t)} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}\cdot e^{\Delta_n\cdot(2\cdot\max_x\{V_t(x;\Delta_n)-V_t(x)\}+\varepsilon-\hat{\varepsilon})\,dy}\,dy\right] \overset{(i)}{=} 0,
\end{aligned}\tag{B.55}$$

where equality (i) follows the fact that $\exists\, N_{\hat{\varepsilon}-\varepsilon}$ such that $\forall\, n \ge N_{\hat{\varepsilon}-\varepsilon}$ such that

$$2\cdot\max_x\{V_t(x;\Delta_n)-V_t(x)\} + V_t(x) - V_t(y)) < 2\cdot\max_x\{V_t(x;\Delta_n)-V_t(x)\} + \varepsilon - \hat{\varepsilon} < 0$$

due to uniform convergence of $V_t(x;\Delta_n) \to V_t(x)$ in x. Moreover, $\frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)}$ is bounded due to the fact that $x \in R(t)$ and, consequently $f_t(x;\Delta_n) > f_t(x)/2 > 0$ for n sufficiently large. Since (B.55) holds for every $\hat{\varepsilon} > \varepsilon$, it implies that

$$\begin{aligned}&\lim_{n\to\infty}\left[\int \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)} \cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy\right]\\ &= \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} \frac{f_t(y;\Delta_n)}{f_t(x;\Delta_n)} \cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy\right].\end{aligned} \tag{B.56}$$

Substitute (B.56) into (B.54) to obtain

$$\begin{aligned}V_t(l_t) &\le \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} V_t(x)\cdot \frac{f_t(x;\Delta_n)}{\int f_t(y;\Delta_n)\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\ &\overset{(i)}{\le} \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} V_t(x)\cdot \frac{f_t(x;\Delta_n)}{\int_{R(t)\setminus Q_{\hat{\varepsilon}}(t)} f_t(y;\Delta_n)\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right].\end{aligned} \tag{B.57}$$

Inequality (B.57) holds for any $\hat{\varepsilon}$, implying that

$$V_t(l_t) \le \lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} V_t(x)\cdot \frac{f_t(x;\Delta_n)}{\int_{R(t)\setminus Q_\varepsilon(t)} f_t(y;\Delta_n)\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]. \tag{B.58}$$

Inequality (B.58) holds for any $\varepsilon > 0$, implying that

$$\begin{aligned}V_t(l_t) &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left[\int_{R(t)\setminus Q_\varepsilon(t)} \frac{V_t(x)\cdot f_t(x;\Delta_n)}{\int_{R(t)\setminus Q_\varepsilon(t)} f_t(y;\Delta_n)\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\ &\overset{(i)}{=} \lim_{n\to\infty}\left[\int_{R(t)\cap\arg\min V_t(x)} \frac{V_t(x)\cdot f_t(x;\Delta_n)}{\int_{R(t)\cap\arg\min V_t(x)} f_t(y;\Delta_n)\cdot e^{\Delta_n\cdot(V_t(x;\Delta_n)-V_t(y;\Delta_n))}\,dy}\,dx\right]\\ &\overset{(ii)}{=} \min V_t(x).\end{aligned} \tag{B.59}$$

where change of limits in (i) is possible since $Q_\varepsilon(t)$ does not depend on n and equality (ii) holds because $V_t(x)$ is convex, and so is the set $\arg\min V_t(x)$. Inequality (B.59) proves that $V_t(l_t) \le \min_{l_t\in R(t)} V_t(l_t)$, which, combined with the requirement that $l_t \in R(t)$ implies that $V_t(l_t) = \min_{l_t\in R(t)} V_t(l_t)$. □

B.5 Lowest Continuation Surplus Equilibrium with Binary Types

In this section we consider a simplified version of the model in which there are only two types, $\tilde{p}_0 \in \{\underline{p}, \overline{p}\} = \{p^L, p^H\}$ and show that there is a unique equilibrium satisfying the lowest continuation surplus refinement 1. As there are only two types, denote by $p_t^i = \pi(p^i, t)$ the posterior belief and $V_t^i = V_t(p^i) - V$ the intermediary's continuation value net of her opportunity cost V for an agent of ex-ante type $i \in \{L, H\}$.

As in Section B.3, define t_S to be the last moment before time $\bar{t} = \sup\{support(\tau)\}$ when a type separates: $t_S \stackrel{def}{=} \sup\{t < \bar{t} : \quad t \in \mathbb{T}\}$. The following Lemma is the analogue of Lemma B.5 for a binary ex-ante type distribution.

Lemma B.15 (Final period separation characterization)**.** *Suppose belief process $l = (l_t)_{t \geq 0}$ is right-continuous and satisfies the lowest continuation surplus refinement 1. Then it must be the case that*

(i) both types find it weakly optimal to separate at $\bar{t}$, i.e., $\bar{t}(p^L) = \bar{t}(p^H) = \bar{t}$;

(ii) there exists a quiet period prior to $\bar{t}$, i.e., $t_S < \bar{t}$;

(iii) clients attribute separations during $[t_S, \bar{t}]$ to low skilled agents, i.e., $l_t = p^L$ for every $t \in (t_S, \bar{t})$.

Proof. Suppose $\bar{t}(p^L) < \bar{t}(p^H)$. This implies that $R(t) = \{p^H\}$ for $t \in (\bar{t}(p^L), \bar{t}(p^H)]$. Following the lowest continuation surplus refinement 1, it follows that $l_t = p^H$ for $t \in (\bar{t}(p^L), \bar{t}(p^H)]$, implying a contradiction with the optimality of separating at time $\bar{t}(p^L)$. A symmetric argument is in effect if $\bar{t}(p^L) > \bar{t}(p^H)$ implying that it has to be the case that $\bar{t}(p^L) = \bar{t}(p^H)$.

Results (*ii*) and (*iii*) follow from the already proven Lemma B.5 of Section B.3. □

We now show that there exists a unique equilibrium satisfying the lowest continuation surplus refinement 1. As the model features only two types, we weaken the clients' belief continuity assumption with right-continuity.

Lemma B.16 (Unique equilibrium)**.** *Suppose belief process $l = (l_t)_{t \geq 0}$ is right continuous and satisfies the lowest continuation surplus refinement 1. Then $l_t = p^L$ for every $t < \bar{t}$.*

Proof. Following Lemma B.15 it follows that $l_t = p^L$ for $t \in (t_S, \bar{t})$. Right continuity of beliefs then implies that $l_{t_S} = p^L$, meaning that it is the L type separating at time t_S. There exists an $\varepsilon > 0$ such that

$$A(\pi(m_{t_S}, t)) - rV - A(p_t^L) < 0 \qquad \forall\, t \in (t_S, t_S + \varepsilon)$$

as, otherwise, type p^L would find it strictly preferable to wait past t_S to separate, which would contradict the definition of t_S. Define

$$t_Q \stackrel{def}{=} \sup\{t < t_S : \qquad \exists \varepsilon > 0 \quad s.t. \quad (t - \varepsilon, t) \notin \mathbb{T}^I\}.$$

Following the same argument and derivations of Lemma B.8 for the continuum of types it follows that there is no pooling during $[t_Q, t_S]$, implying that $l_t = p^L$ for all $t \in [t_Q, t_S]$. Since it's the same type p^L that is separating during the period $[t_Q, t_S]$, it implies that

$$A(\pi(m_t, t)) - rV - A(p_t^L) = 0 \qquad \forall\, t \in [t_Q, t_S].$$

and that there exists an $\varepsilon > 0$ such that

$$A(\pi(m_t, t)) - rV - A(p_t^L) > 0 \qquad \forall\, t \in (t_Q - \varepsilon, t_Q). \tag{B.60}$$

The next step is to show that type p^H finds it strictly preferable to wait until time $\bar{t}$ to separate than to separate at time t_Q and belief p^L. Denote by $L_t(p)$ the belief at which type p is willing to separate at time t. It satisfies (B.13) which we rewrite here as

$$V + U_t(L_t(p)) = W_t(p).$$

for $p \in \{p^L, p^H\}$. At $t = \bar{t}$ both types are willing to separate at the pooling belief, implying that $L_{\bar{t}}(p^L) = L_{\bar{t}}(p^H) = l_{\bar{t}}$. It follows that

$$\dot{L}_t(p) = \frac{A(\pi(L_t(p), t)) + rV - A(\pi(m_t, t))}{\partial_1 \pi(L_t(p), t) \,\cdot\, \partial_2 U(\pi(p, t), \pi(L_t(p), t))}.$$

Types p^L and p^H are willing to pool only if there indifference beliefs are equal, i.e., types p^L and p^H find pooling at time $\hat{t}$ incentive compatible if and only if $L_{\hat{t}}(p^L) = L_{\hat{t}}(p^H) = l_{\hat{t}}$. For $t \in [t_S, \bar{t}]$ we have $\dot{L}_t(p^H) < \dot{L}_t(p^L)$, while for $t \in [t_Q, t_S]$ we have $\dot{L}_t(p^H) < \dot{L}_t(p^L) = 0$. Consequently, $L_{t_Q}(p^H) > L_{t_Q}(p^L)$, implying that $V_{t_Q}(p^H) > V_{t_Q}(p^L)$.

Suppose there exists a time $t_* \in [0, t^*)$ which corresponds to the last time when any type is willing to separate preceding time t_Q, i.e.,

$$t_* = \sup\{t < t_Q : \quad t \in \mathbb{T}^I\}.$$

By definition of t_Q, it must be the case that $t_* < t_Q$. Due to the linearity of the value function $V_t(p)$ in p, the local arguments in Lemma B.15, inherited from Lemma B.5, extend to the quiet period (t^*, t_Q), implying that $l_t = p^L$ for $t \in (t^*, t_Q)$. Right continuity of beliefs then requires that $l_{t^*} = p^L$. Since period (t^*, t_Q) is a quiet period, it follows from (B.60) that

$$A(\pi(m_{t^*}, t^*)) - rV - A(\pi(p^L, t^*)) > 0.$$

This leads to a contradiction with the existence of t^* as the L type manager would strictly benefit from waiting past t^* to separate. Consequently there are no separations in equilibrium during $[0, t_Q)$.

Finally, we show that that $l_t = p^L$ for all $t < \bar{t}$. The comparison $V_t^H > V_t^L$ follows from

$$V_t^H = \int_t^{t_Q} e^{-r(s-t)} \left(p_t^H + (1 - p_t^H) e^{-\lambda(s-t)}\right) \underbrace{\left(A(\pi(m_s, s)) - rV - A(p_s^L)\right)}_{>0} ds$$

$$
\begin{aligned}
&+\int_{t_Q}^{t_S} e^{-r(s-t)}\left(p_t^H+(1-p_t^H)e^{-\lambda(s-t)}\right)\underbrace{\left(A(\pi(m_s,s))-rV-A(p_s^L)\right)}_{=0} ds\\
&+\int_{t_Q}^{\infty} e^{-r(s-t)}\left(p_t^H+(1-p_t^H)e^{-\lambda(s-t)}\right)\left(A(\pi(m_{\bar{t}},s))-A(p_s^L)\right) ds\\
&\overset{(i)}{>}\int_{t}^{t_Q} e^{-r(s-t)}\left(p_t^H+(1-p_t^H)e^{-\lambda(s-t)}\right)\underbrace{\left(A(\pi(m_s,s))-rV-A(p_s^L)\right)}_{\geq 0} ds\\
&>\int_{t}^{t_Q} e^{-r(s-t)}\left(p_t^L+(1-p_t^L)e^{-\lambda(s-t)}\right)\left(A(\pi(m_s,s))-rV-A(p_s^L)\right) ds=V_t^L,
\end{aligned}
$$

where the strict inequality (i) follows from $V_{t_Q}(p^H)>V_{t_Q}(p^L)$. □

B.6 Perturbed Model Equilibrium Construction with a Binary Type

We now consider the perturbation of the model, as described in Section B.4 of this Online Appendix B, in which the intermediary-agent pair receives a relationship-specific disutility shock distributed exponentially with parameter Δ. In Section B.6.2 we explicitly construct equilibria for a given parameter Δ of the exponential shock distribution and show that this sequence of equilibria features a equilibrium limit as $\Delta\to\infty$. We then show in Section B.6.3 that any such equilibrium limit has to satisfy the lowest continuation surplus refinement 1 and, consequently, converge to the equilibrium characterized in Section B.5. As the shocks follow an exponential distribution, the probability that a shock exceeds a value x is equal to $G(x)=e^{-\Delta\cdot x}$.

B.6.1 Basic Properties of the Model with Shocks and Binary Types

Denote by e_t^L and e_t^H to be the rate at which the intermediary lets go of the agent of ex-ante ability p^L and p^H respectively at time t. If $V_t^i>0$, then the intermediary prefers to retain the agent absent any shocks, consequently $e_t^i=\varepsilon\cdot G(V_t^i)$. If, however, $V_t^i=0$, then the intermediary lets the agent go with intensity ε if any shock arrives, but then may further churn the agent for strategic considerations. Consequently, $e_t^i\geq\varepsilon$ if $V_t^i=0$.

Lemma B.17. *The probability $\alpha_t=\mathrm{P}_t\big(p=\tilde{p}^H\,|\,X_t=t,\tau>t\big)$ that the remaining type is a high type conditional on good performance and continued employment by the intermediary satisfies*

$$
\dot{\alpha}_t=\alpha_t(1-\alpha_t)\cdot\left(e_t^L-e_t^H+\lambda\cdot\left(p_t^H-p_t^L\right)\right). \tag{B.61}
$$

Proof. Denote by α_t the fraction of low types that survive. Then

$$\alpha_{t+dt} = \mathrm{P}\big(\tilde{p}_t = p_t^H \,|\, remain\big) = \frac{\mathrm{P}\big(\tilde{p}_t = p_t^H,\, remain\big)}{\mathrm{P}\big(\tilde{p}_t = p_t^H,\, remain\big) + \mathrm{P}\big(\tilde{p}_t = p_t^L,\, remain\big)}$$
$$= \frac{\alpha_t \cdot (1 - e_t^H\, dt) \cdot (1 - (1 - p_t^H)\lambda\, dt)}{\alpha_t \cdot (1 - e_t^H\, dt) \cdot (1 - (1 - p_t^H)\lambda\, dt) + (1 - \alpha_t) \cdot (1 - e_t^L\, dt) \cdot (1 - (1 - p_t^L)\lambda\, dt)}.$$

Then

$$\alpha_{t+dt} - \alpha_t = \frac{\alpha_t \cdot (1 - e_t^H\, dt) \cdot (1 - (1 - p_t^H)\lambda\, dt)}{\alpha_t \cdot (1 - e_t^H\, dt) \cdot (1 - (1 - p_t^H)\lambda\, dt) + (1 - \alpha_t) \cdot (1 - e_t^L\, dt) \cdot (1 - (1 - p_t^L)\lambda\, dt)} - \alpha_t$$
$$= \frac{\alpha_t(1 - \alpha_t) \cdot \big(e_t^L - \lambda \cdot p_t^L - e_t^H + \lambda \cdot p_t^H\big) \cdot dt}{\big(\alpha_t \cdot (1 - e_t^H\, dt - (1 - p_t^H)\lambda\, dt) + (1 - \alpha_t) \cdot (1 - e_t^L\, dt - (1 - p_t^L)\lambda\, dt)\big)^2}.$$

Dividing both sides by dt and taking $dt = 0$ obtain (B.61). □

In what follows, we construct an equilibrium in which prior to the final date $\bar{t}$ both types separate continuously for $t < \bar{t}$ and may feature an atom of separations in the last period $t = \bar{t}$. In other words, we construct an equilibrium in which separation rates e_t^i are finite for all $t < \bar{t}$. In this case, the average separating type k_t along the path of good performance is given by

$$k_t = \frac{p_t^L \cdot (1 - \alpha_t) \cdot e_t^L + p_t^H \cdot \alpha_t \cdot e_t^H}{(1 - \alpha_t) \cdot e_t^L + \alpha_t \cdot e_t^H}. \tag{B.62}$$

Lemma B.18 (Monotone equilibrium)**.** *Suppose $\varepsilon < \bar{\varepsilon}$. Then, in equilibrium it must be the case that $k_t \le q_t$ with the inequality being strict for any $t < \bar{t}$. Consequently, $V_t^H \ge V_t^L$ and $\mathbb{T}^H \subseteq \mathbb{T}(p^L)$.*

Proof. Without loss of generality, suppose $\bar{t} > 0$ and $k_0 \ge q_0$. Define $\mathbb{T}\left(()\, p\right)$ as the set of times when the intermediary chooses to let go of the agent in the absence of a shock. Define

$$\hat{t} \overset{def}{=} \inf \left\{ t > 0 :\; k_t \in \mathbb{T}\left(p^L\right) \cup \mathbb{T}\left(p^H\right) \text{ and } k_t \le q_t \right\}.$$

The time $\hat{t}$ is well defined as $k_{\bar{t}} = q_{\bar{t}}$. There exists an ε sufficiently low so that $q_t - \frac{rV}{2} < \pi(k_0, t)$ for every $t \in (0, \bar{t})$ since the types that are departing voluntarily prior to $\hat{t}$ are better than q_t, thus lowering the average, while the rate of exogenous departures ε can be made sufficiently small relative to $r \cdot V$. Define $\hat{\alpha}_t$ as the fraction of high types if (i) there are no departures in the absence of a shock, (ii) all low types leave following arrival of a shock, and (iii) no high type leaves following a shock. Then, $\hat{\alpha}_t \ge \alpha_t$ and from Lemma B.17 it follows that

$$\dot{\hat{\alpha}}_t = \hat{\alpha}_t \left(1 - \hat{\alpha}_t\right) \cdot \left[\varepsilon + \lambda \cdot \left(p_t^H - p_t^L\right)\right]$$

$$\frac{d}{dt}\ln\left(\frac{\hat{\alpha}_t}{1-\hat{\alpha}_t}\right) = \varepsilon + \lambda\cdot(p_t^H - p_t^L)$$
$$\Rightarrow \qquad \hat{\alpha}_t = \frac{\alpha_0\cdot e^{\varepsilon\cdot t+\lambda\int_0^t(p_s^H-p_s^L)\,ds}}{1-\alpha_0+\alpha_0\cdot e^{\varepsilon\cdot t+\lambda\int_0^t(p_s^H-p_s^L)\,ds}}.$$

This, then implies that

$$q_t = (1-\alpha_t)\cdot p_t^L + \alpha_t\cdot p_t^H \le (1-\hat{\alpha}_t)\cdot p_t^L + \hat{\alpha}_t\cdot p_t^H. \tag{B.63}$$

The difference between q_t and $\pi(q_0,t)$ is given by

$$\begin{aligned}
q_t - \pi(q_0,t)) &\le (1-\hat{\alpha}_t)\cdot p_t^L + \hat{\alpha}_t\cdot p_t^H - \pi(q_0,t)\\
&\le \left(\frac{\alpha_0\cdot e^{\varepsilon\cdot t+\lambda\int_0^t(p_s^H-p_s^L)\,ds}}{1-\alpha_0+\alpha_0\cdot e^{\varepsilon\cdot t+\lambda\int_0^t(p_s^H-p_s^L)\,ds}} - \frac{\alpha_0\cdot e^{\lambda\int_0^t(p_s^H-p_s^L)\,ds}}{1-\alpha_0+\alpha_0\cdot e^{\lambda\int_0^t(p_s^H-p_s^L)\,ds}}\right)\cdot(p_t^H-p_t^L)\\
&= \frac{\alpha_0(1-\alpha_0)\cdot e^{\lambda\int_0^t(p_s^H-p_s^L)\,ds}\cdot\left(e^{\varepsilon t}-1\right)}{\left(1-\alpha_0+\alpha_0\cdot e^{\varepsilon\cdot t+\lambda\int_0^t(p_s^H-p_s^L)\,ds}\right)\left(1-\alpha_0+\alpha_0\cdot e^{\lambda\int_0^t(p_s^H-p_s^L)\,ds}\right)}\cdot(p_t^H-p_t^L)\\
&< (e^{\varepsilon t}-1)\cdot(p_t^H-p_t^L).
\end{aligned} \tag{B.64}$$

In order for separation at date $\hat{t}$ to be incentive compatible it must be the case that

$$\begin{aligned}
&\int_0^{\hat{t}} e^{-rt}\cdot\left(p+(1-p)\cdot e^{-\lambda t}\right)\cdot(A(q_t)-rV)\,dt\\
&+\int_{\hat{t}}^{\infty} e^{-rt}\cdot\left(p+(1-p)\cdot e^{-\lambda t}\right)\cdot A\left(\pi(k_{\hat{t}},t-\hat{t})\right)\,dt\\
&\ge \int_0^{\infty} e^{-rt}\cdot\left(p+(1-p)\cdot e^{-\lambda t}\right)\cdot A(\pi(k_0,t))\,dt.
\end{aligned} \tag{B.65}$$

A necessary condition for (B.65) to hold is if it holds at $k_{\hat{t}} = q_{\hat{t}}$ and $k_0 = q_0$. Substituting these values into (B.65) obtain

$$\begin{aligned}
&\int_{\hat{t}}^{\infty} e^{-rt}\cdot\left(p+(1-p)\cdot e^{-\lambda t}\right)\cdot\left(A\left(\pi(q_{\hat{t}},t-\hat{t})\right)-A(\pi(q_0,t))\right)dt\\
&\ge \int_0^{\hat{t}} e^{-rt}\cdot\left(p+(1-p)\cdot e^{-\lambda t}\right)\cdot\underbrace{\left(rV+A(\pi(q_0,t))-A(q_t)\right)}_{\ge r\cdot V/2}\,dt\\
&\ge \int_0^{\hat{t}} e^{-rt}\cdot\left(p+(1-p)\cdot e^{-\lambda t}\right)\cdot\frac{rV}{2}\,dt.
\end{aligned} \tag{B.66}$$

The upper bound obtained in (B.64) ensures that inequality (B.66) cannot be satisfied for $\varepsilon \le \bar{\varepsilon}$ uniformly across Δ. □

Lemma B.19. *Suppose separation rates* $(e_t^L, e_t^H)_{t\geq 0}$ *are differentiable almost everywhere. Then the churning rate* γ_t*, as defined in* (12) *is given by*

$$\gamma_t \stackrel{def}{=} \dot{k}_t - \lambda k_t(1-k_t) = \frac{\alpha_t(1-\alpha_t)\cdot\left(p_t^H - p_t^L\right)}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t \cdot e_t^H\right)^2}\cdot\left(\dot{e}_t^H \cdot e_t^L - \dot{e}_t^L \cdot e_t^H\right). \tag{B.67}$$

Proof. If e_t^L and e_t^H are kept constant, i.e., the fractions of types separating are constant, then it follows that $\dot{k}_t = \lambda k_t(1-k_t)$ and $\gamma_t = 0$. Consequently,

$$\begin{aligned}
\gamma_t =& \frac{p_t^L\cdot(1-\alpha_t)\cdot\dot{e}_t^L + p_t^H\cdot\alpha_t\cdot\dot{e}_t^H}{(1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H} - \frac{p_t^L\cdot(1-\alpha_t)\cdot e_t^L + p_t^H\cdot\alpha_t\cdot e_t^H}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2}\cdot\left((1-\alpha_t)\cdot\dot{e}_t^L + \alpha_t\cdot\dot{e}_t^H\right)\\
=& \frac{\dot{e}_t^L\cdot(1-\alpha_t)\cdot\left[p_t^L\cdot(1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H - p_t^L\cdot(1-\alpha_t)\cdot e_t^L - p_t^H\cdot\alpha_t\cdot e_t^H\right]}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2}\\
&+ \frac{\dot{e}_t^H\cdot\alpha_t\cdot\left[p_t^H\cdot\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right) - p_t^L\cdot(1-\alpha_t)\cdot e_t^L - p_t^H\cdot\alpha_t\cdot e_t^H\right]}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2}\\
=& \frac{\dot{e}_t^L\cdot(1-\alpha_t)\cdot\left[p_t^L\cdot\alpha_t\cdot e_t^H - p_t^H\cdot\alpha_t\cdot e_t^H\right]}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2} + \frac{\dot{e}_t^H\cdot\alpha_t\cdot\left[p_t^H\cdot(1-\alpha_t)\cdot e_t^L - p_t^L\cdot(1-\alpha_t)\cdot e_t^L\right]}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2}\\
=& \frac{\dot{e}_t^H\cdot\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot e_t^L - \dot{e}_t^L\cdot\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot e_t^H}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2}\\
=& \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\left[\dot{e}_t^H\cdot e_t^L - \dot{e}_t^L\cdot e_t^H\right]}{\left((1-\alpha_t)\cdot e_t^L + \alpha_t\cdot e_t^H\right)^2}.
\end{aligned}$$

□

If both $V_t^L > 0$ and $V_t^H > 0$ then (B.67) becomes

$$\begin{aligned}
\gamma_t &= \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\left[-\varepsilon^2\cdot g(V_t^H)\cdot G(V_t^L)\cdot\dot{V}_t^H + \varepsilon^2\cdot g(V_t^L)\cdot G(V_t^H)\cdot\dot{V}_t^L\right]}{\left((1-\alpha_t)\cdot\varepsilon\cdot G(V_t^L) + \alpha_t\cdot\varepsilon\cdot G(V_t^H)\right)^2}\\
&= \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)}{\left((1-\alpha_t)\cdot G(V_t^L) + \alpha_t\cdot G(V_t^H)\right)^2}\cdot\left[g(V_t^L)\cdot G(V_t^H)\cdot\dot{V}_t^L - g(V_t^H)\cdot G(V_t^L)\cdot\dot{V}_t^H\right],
\end{aligned}$$

where $g(\cdot) = \Delta\cdot e^{-\Delta x}$ is the pdf of the shock distribution. If $V_t^H > 0$ while $V_t^L = 0$ then (B.67) becomes

$$\gamma_t = \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)}{\left((1-\alpha_t)\cdot(1+e_t^L) + \alpha_t\cdot G(V_t^H)\right)^2}\cdot\left[G(V_t^H)\cdot(g(0)\cdot\dot{V}_t^L - \dot{e}_t^L) - g(V_t^H)\cdot(1+e_t^L)\cdot\dot{V}_t^H\right].$$

Under continuous separations the value functions of type i satisfies

$$
\begin{aligned}
rV_t^i = {} & A(q_t) - A(k_t) - rV + \gamma_t \cdot \partial_2 U(p_t^i, k_t) - \lambda \cdot (1 - p_t^i) \cdot V_t^i + \dot{V}_t^i \\
& - \varepsilon \cdot G(-V_t^i) \cdot V_t^i + \varepsilon \cdot \mathrm{E}\left[\tilde{s} \cdot \mathbb{1}\left\{\tilde{s} \geq -V_t^i\right\}\right] + \max_{e_t^i \geq 0}\{-e_t^i \cdot V_t^i\},
\end{aligned} \tag{B.68}
$$

B.6.2 Candidate Equilibrium Construction with a Binary Type

Consider a time $\bar{t}$ and suppose at $\bar{t}$ both types pool and leave the intermediary at a separating belief is $\pi(m, \bar{t})$, where $m > q_0$.[7]

Boundary conditions. At $t = \bar{t}-$ the continuation value function $V_{\bar{t}-}^i$ of type i and the average separating type $k_{\bar{t}-}$ are given by

$$
\begin{aligned}
V_{\bar{t}-}^i &= u\Big(p_{\bar{t}}^i, \pi(m, \bar{t})\Big) - u\Big(p_{\bar{t}}^i, k_{\bar{t}-}\Big), \qquad i \in \{L, H\}, \\
k_{\bar{t}-} &= \frac{p_{\bar{t}}^L \cdot (1 - \alpha_{\bar{t}}) \cdot G\left(V_{\bar{t}-}^L\right) + p_{\bar{t}}^H \cdot \alpha_{\bar{t}} \cdot G\left(V_{\bar{t}-}^H\right)}{(1 - \alpha_{\bar{t}}) \cdot G\left(V_{\bar{t}-}^L\right) + \alpha_{\bar{t}} \cdot G\left(V_{\bar{t}-}^H\right)}.
\end{aligned} \tag{B.69}
$$

As shocks are exponentially distributed, $G(x) = e^{-\Delta x}$. Substituting $V_{\bar{t}-}^i$ into the expression for $k_{\bar{t}}$ obtain

$$
\begin{aligned}
k_{\bar{t}-} &= \frac{p_{\bar{t}}^L \cdot (1 - \alpha_{\bar{t}}) + p_{\bar{t}}^H \cdot \alpha_{\bar{t}} \cdot e^{\Delta\left(u\left(p_{\bar{t}}^L, \pi(m,\bar{t})\right) - u\left(p_{\bar{t}}^L, k_{\bar{t}-}\right) - u\left(p_{\bar{t}}^H, \pi(m,\bar{t})\right) + u\left(p_{\bar{t}}^H, k_{\bar{t}-}\right)\right)}}{(1 - \alpha_{\bar{t}}) + \alpha_{\bar{t}} \cdot e^{\Delta\left(u\left(p_{\bar{t}}^L, \pi(m,\bar{t})\right) - u\left(p_{\bar{t}}^L, k_{\bar{t}-}\right) - u\left(p_{\bar{t}}^H, \pi(m,\bar{t})\right) + u\left(p_{\bar{t}}^H, k_{\bar{t}-}\right)\right)}} \\
&= \frac{p_{\bar{t}}^L \cdot (1 - \alpha_{\bar{t}}) + p_{\bar{t}}^H \cdot \alpha_{\bar{t}} \cdot e^{\Delta \cdot (p_{\bar{t}}^L - p_{\bar{t}}^H) \cdot \left(u_1(\pi(m,\bar{t})) - u_1(k_{\bar{t}-}) - u_0(\pi(m,\bar{t})) + u_0(k_{\bar{t}-})\right)}}{(1 - \alpha_{\bar{t}}) + \alpha_{\bar{t}} \cdot e^{\Delta \cdot (p_{\bar{t}}^L - p_{\bar{t}}^H) \cdot \left(u_1(\pi(m,\bar{t})) - u_1(k_{\bar{t}-}) - u_0(\pi(m,\bar{t})) + u_0(k_{\bar{t}-})\right)}} \\
&= \frac{p_{\bar{t}}^L \cdot (1 - \alpha_{\bar{t}}) \cdot e^{\Delta \cdot (p_{\bar{t}}^H - p_{\bar{t}}^L) \cdot \left(u_1(\pi(m,\bar{t})) - u_1(k_{\bar{t}-}) - u_0(\pi(m,\bar{t})) + u_0(k_{\bar{t}-})\right)} + p_{\bar{t}}^H \cdot \alpha_{\bar{t}}}{(1 - \alpha_{\bar{t}}) \cdot e^{\Delta \cdot (p_{\bar{t}}^H - p_{\bar{t}}^L) \cdot \left(u_1(\pi(m,\bar{t})) - u_1(k_{\bar{t}-}) - u_0(\pi(m,\bar{t})) + u_0(k_{\bar{t}-})\right)} + \alpha_{\bar{t}}} \\
&= p_{\bar{t}}^L + \frac{\left(p_{\bar{t}}^H - p_{\bar{t}}^L\right) \cdot \alpha_{\bar{t}}}{(1 - \alpha_{\bar{t}}) \cdot e^{\Delta \cdot (p_{\bar{t}}^H - p_{\bar{t}}^L) \cdot \left(u_1(\pi(m,\bar{t})) - u_1(k_{\bar{t}-}) - u_0(\pi(m,\bar{t})) + u_0(k_{\bar{t}-})\right)} + \alpha_{\bar{t}}}.
\end{aligned} \tag{B.70}
$$

There are at least two solutions to equation (B.70). Consider the smallest one of them.

Lemma B.20. *The smallest solution $k_{\bar{t}-}$ to* (B.70) *converges to $p_{\bar{t}}^L$ as $\Delta \to \infty$.*

[7] For a given $m > q_0$ we will solve for time $\bar{t}$ that is consistent with the average type being equal to $\pi(m, \bar{t})$ at time $\bar{t}$.

Proof. The derivative of the right hand side of (B.70) with respect to k is

$$\frac{\left(p_{\bar t}^H - p_{\bar t}^L\right) \cdot \alpha_{\bar t} \cdot \Delta \cdot e^{\Delta \cdot (p_{\bar t}^H - p_{\bar t}^L) \cdot \left[u_1(\pi(m,\bar t)) - u_1(k_{\bar t -}) - u_0(\pi(m,\bar t)) + u_0(k_{\bar t -})\right]} \cdot \left[u_1'(k_{\bar t -}) - u_0'(k_{\bar t -})\right]}{\left[(1-\alpha_{\bar t}) \cdot e^{\Delta \cdot (p_{\bar t}^H - p_{\bar t}^L) \cdot \left[u_1(\pi(m,\bar t)) - u_1(k_{\bar t -}) - u_0(\pi(m,\bar t)) + u_0(k_{\bar t -})\right]} + \alpha_{\bar t}\right]^2}$$

This derivative is strictly positive. At $k_{\bar t -} = \pi(m, \bar t)$ this derivative is equal to

$$\left(p_{\bar t}^H - p_{\bar t}^L\right) \cdot \alpha_{\bar t} \cdot \Delta \cdot \left[u_1'(k_{\bar t -}) - u_0'(k_{\bar t -})\right]$$

which is strictly greater than 1 for Δ being sufficiently large.

Consider a candidate $k = \alpha \cdot p_{\bar t}^L + (1-\alpha) \cdot \pi(m, \bar t)$. At this level there exists a sufficiently high Δ such that the right hand side is approximately equal to $p_{\bar t}^L$. This means that the smallest solution is strictly less than $\frac{p_{\bar t}^L + \pi(m,\bar t)}{2}$. As $\Delta \to \infty$ it follows that $k_t \to p_{\bar t}^L$. □

Corollary B.1. *For $k_{\bar t -}$ being the smallest solution to* (B.70)*, the value functions $V_t^i > 0$ for t in the vicinity of $\bar t$ for both types $i \in \{L, H\}$.*

Quiet period dynamics. Fix $\bar v > 0$. From the boundary condition $q_{\bar t} = \pi(m, \bar t)$, which is equivalent to $\alpha_{\bar t} = \frac{\pi(m,\bar t) - p_{\bar t}^L}{p_{\bar t}^H - p_{\bar t}^L}$, and $k_{\bar t -}$ being the smallest solution to (B.70), and value function boundary values $V_{\bar t -}^i$ being pinned down by (B.69), solve backward via a system of first-order differential equations:

$$\begin{cases} rV_t^H = A(q_t) - A(k_t) - rV + \gamma_t \cdot \partial_2 U(p_t^H, k_t) - \lambda(1-p_t^H) \cdot V_t^H + \dot V_t^H - \varepsilon \cdot \mathrm{E}\left[\min\{\tilde s, V_t^H\}\right], \\ \hat V_t^H = \max\left\{V_t^L + \bar v, \quad V_t^H\right\}, \\ rV_t^L = A(q_t) - A(k_t) - rV + \gamma_t \cdot \partial_2 U(p_t^L, k_t) - \lambda(1-p_t^L) \cdot V_t^L + \dot V_t^L - \varepsilon \cdot \mathrm{E}\left[\min\{\tilde s, V_t^L\}\right], \\ k_t = \dfrac{p_t^L \cdot (1-\alpha_t) \cdot G(V_t^L) + p_t^H \cdot \alpha_t \cdot G\left(\hat V_t^H\right)}{(1-\alpha_t) \cdot G(V_t^L) + \alpha_t \cdot G\left(\hat V_t^H\right)}, \\ \gamma_t = \dfrac{\alpha_t(1-\alpha_t) \cdot (p_t^H - p_t^L) \cdot \left[G\left(\hat V_t^H\right) \cdot g(V_t^L) \cdot \dot V_t^L - G(V_t^L) \cdot g\left(\hat V_t^H\right) \cdot \dot{\hat V}_t^H\right]}{\left[(1-\alpha_t) \cdot G(V_t^L) + \alpha_t \cdot G\left(\hat V_t^H\right)\right]^2}, \\ \dot\alpha_t = \alpha_t(1-\alpha_t) \cdot \left[\varepsilon \cdot \left(G(V_t^L) - G\left(\hat V_t^H\right)\right) + \lambda \cdot (p_t^H - p_t^L)\right]. \end{cases} \tag{B.71}$$

Denote by $t_1 \overset{def}{=} \sup\{t < \bar t : \quad V_t^L = 0\}$ to be the first time prior to $\bar t$ when $V_t^L = 0$. Since $V_t^L > 0$ for $t \in (t_1, \bar t]$ it follows that $\dot V_{t_1}^L \geq 0$. Using the fact that $G(x) = e^{-\Delta x}$ and $g(x) = \Delta \cdot e^{-\Delta x}$ express γ_t as

$$\gamma_t = \frac{\alpha_t(1-\alpha_t) \cdot (p_t^H - p_t^L) \cdot \Delta \cdot e^{\Delta\left(\hat V_t^H - V_t^L\right)}}{\left[(1-\alpha_t) \cdot e^{\Delta\left(\hat V_t^H - V_t^L\right)} + \alpha_t\right]^2} \cdot \left[\dot V_t^L - \dot{\hat V}_t^H\right]. \tag{B.72}$$

Consider two cases.

(i) Suppose $V_t^H \geq V_t^L + \bar{v}$. Then $\hat{V}_t^H = V_t^H$ and the Jacobian matrix for the system of ODEs in (B.71) can be written as

$$J_t = \begin{bmatrix} 1 - \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot\partial_2 U(p_t^H, k_t) & \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot\partial_2 U(p_t^H, k_t) \\ -\frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot\partial_2 U(p_t^L, k_t) & 1 + \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot\partial_2 U(p_t^L, k_t) \end{bmatrix}.$$

The determinant of the Jacobian J_t is equal to

$$\begin{aligned} det(J_t) &= 1 + \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot\partial_2 U(p_t^L, k_t) \\ &\quad - \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot\partial_2 U(p_t^H, k_t) \\ &= 1 - \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta(V_t^H - V_t^L)}}{\left((1-\alpha_t)\cdot e^{\Delta(V_t^H - V_t^L)}+\alpha_t\right)^2}\cdot(p_t^H - p_t^L)\cdot(u_1'(k_t) - u_0'(k_t)) \\ &\overset{(i)}{>} 1 - \frac{\alpha_t(1-\alpha_t)\cdot(p_t^H - p_t^L)\cdot\Delta\cdot e^{\Delta\bar{v}}}{((1-\alpha_t)\cdot e^{\Delta\bar{v}}+\alpha_t)^2}\cdot(p_t^H - p_t^L)\cdot(u_1'(k_t) - u_0'(k_t)) \overset{(ii)}{>} 0, \end{aligned}$$

where inequality (i) is satisfied if $\Delta > \frac{1}{\bar{v}}\cdot\ln\left(\frac{\alpha_t}{1-\alpha_t}\right)$ and inequality (ii) is satisfied for Δ being sufficiently large regardless of other parameters, given that $t \leq \bar{T}$.

(ii) Suppose $V_t^H \leq V_t^L + \bar{v}$. In this case $\hat{V}_t^H = V_t^L + \bar{v}$ and from (B.72) it follows that $\gamma_t = 0$. In this case the Jacobian for the system of ODEs is simply equal to

$$J_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \det(J_t) = 1.$$

Churning period dynamics. For $t < t_1$ construct a separation process e_t^L satisfying boundary condition $e_{t_1}^L = 0$ as

$$\begin{cases} rV_t^H = A(q_t) - A(k_t) - rV + \gamma_t \partial_2 U(p_t^H, k_t) - \lambda(1 - p_t^H)V_t^H + \dot{V}_t^H - \varepsilon \cdot \mathrm{E}\left[\min\{\tilde{s}, V_t^H\}\right], \\ \quad 0 = A(q_t) - A(k_t) - rV + \gamma_t \cdot \partial_2 U(p_t^L, k_t), \\ \quad k_t = \dfrac{p_t^L \cdot (1 - \alpha_t) \cdot (1 + e_t^L/\varepsilon) + p_t^H \cdot \alpha_t \cdot G(V_t^H)}{(1 - \alpha_t) \cdot (1 + e_t^L/\varepsilon) + \alpha_t \cdot G(V_t^H)}, \\ \quad \gamma_t = \dfrac{\alpha_t(1 - \alpha_t) \cdot (p_t^H - p_t^L) \cdot \left[-G(V_t^H) \cdot \dot{e}_t^L/\varepsilon - (1 + e_t^L/\varepsilon) \cdot g(V_t^H) \cdot \dot{V}_t^H\right]}{\left((1 - \alpha_t)(1 + e_t^L/\varepsilon) + \alpha_t \cdot G(V_t^H)\right)^2}, \\ \quad \dot{\alpha}_t = \alpha_t(1 - \alpha_t) \cdot \left[\varepsilon + e_t^L - \varepsilon \cdot G(V_t^H) + \lambda \cdot (p_t^H - p_t^L)\right]. \end{cases} \tag{B.73}$$

The system of differential equations (B.73) has no singular points since we can use the expression for γ_t to solve for $\dot{V}_t^H$, and then substitute that expression to obtain e_t^L. At t_1 we set $e_{t_1}^L = 0$, it implies that k_t is continuous at $t = t_1$. Moreover, since $V_{t_1}^L = 0$ and $\dot{V}_{t_1+}^L \geq 0$ it follows that

$$\gamma_{t_1-} = \frac{A(k_{t_1}) + rV - A(q_{t_1})}{\partial_2 U(p_{t_1}^L, k_{t_1})} > \frac{A(k_{t_1}) + rV - A(q_{t_1}) - \dot{V}_{t_1+}^L}{\partial_2 U(p_{t_1}^L, k_{t_1})} = \gamma_{t_1+}.$$

Plugging this relation into the expressions for γ_{t_1-} and γ_{t_1+} obtain

$$\begin{aligned} -G(V_{t_1}^H) \cdot \dot{e}_{t_1}^L/\varepsilon - g(V_{t_1}^H) \cdot \dot{V}_{t_1}^H &\geq G(V_{t_1}^H) \cdot g(0) \cdot \dot{V}_{t_1+}^L - g(V_{t_1}^H) \cdot \dot{V}_{t_1}^H, \\ \Rightarrow \qquad \dot{e}_{t_1}^L &\leq -\varepsilon \cdot g(0) \cdot \dot{V}_{t_1+}^L \leq 0. \end{aligned}$$

This implies that at time t_1 we have $e_{t_1}^L = 0$ and $\dot{e}_{t_1}^L \leq 0$. Continue solving this differential system backward until time $t_2 \overset{def}{=} \sup\{t < t_1 : \quad e_t^L = 0\}$ when the L type need not separate voluntarily. From t_2 continue the process back using the boundary condition $V_{t_2}^L = 0$ via the differential equation system (B.71). Given that $\dot{e}_{t_2+}^L \geq 0$, it follows that $\dot{V}_{t_2-}^L \leq 0$.

Induction step. Continue the process above until time $t = 0$ by switching between differential equation systems (B.71) and (B.73). Denote by $\mathbb{T}$ to be the set of times during which the system follows (B.73). By solving this differential system from $\bar{t}$ to) we obtain some value of q_0, which may differ from the model's given prior, due to our arbitrary choice of a boundary condition $(\bar{t}, \pi(m, \bar{t}))$, and which we will denote by $Q(\bar{t}, m)$.

Lemma B.21 (Low type optimal stopping)**.** *Given belief process $(q_t, k_t)_{t \in [0, \bar{t}]}$ function V_t^L is a solution* *to the optimal stopping problem net of* $U(p_t^L, k_t)$*, i.e.,*

$$V_t^L = \max_{\tau \in [t, \bar{t}(m)]} \left\{ \int_t^\tau e^{-r(s-t)} \cdot \left(p_t^L + (1-p_t^L) \cdot e^{-\lambda t} \right) \cdot (A(q_s) - rV) \, ds \right.$$
$$\left. + e^{-r(\tau - t)} \cdot \left((1-p_t^L) \cdot e^{-\lambda(\tau - t)} \cdot u_0(k_t) + p_t^L \cdot u_1(k_t) \right) \right\} - U(p_t^L, k_t).$$

Every time, including $\bar{t}(m)$ *that takes place during the churning phase* (B.73)*, is an optimal stopping time for the* L *type.*

Proof. Given process (q_t, k_t), value function V_t^L satisfies

$$rV_t^L = A(q_t) - A(k_t) - rV + \gamma_t \cdot \partial_2 U(p_t^L, k_t) - \lambda(1-p_t^L) \cdot V_t^L + \dot{V}_t^L - \varepsilon \cdot \mathrm{E}\left[\min\{\tilde{s}, V_t^L\}\right]$$

during the waiting period. For t in the exercise period $V_t^L = 0$ it follows that

$$0 = A(q_t) - A(k_t) - rV + \gamma_t \partial_2 U(p_t^L, k_t).$$

These ordinary differential equations are equivalent to those used to construct V_t^L via (B.71) and (B.73). □

Corollary B.2. *For* $\varepsilon < \bar{\varepsilon}$ *it must be the case that* $k_t \leq q_t$ *for* $t < \bar{t}$*.*

Proof. If this were not the case, then there would be a contradiction with Lemma B.21 of optimal stopping at time $\bar{t}$ for type L, following from Lemma B.18. □

Lemma B.22 (Existence of $\bar{t}$)**.** *There exists a* $\bar{t} \leq \bar{T}$ *such that for any* $m > q_0$ *we have* $q_0 = Q(\bar{t}, m)$*.*

Proof. At $\bar{t} = 0$ we have $Q(m, \bar{t}) = q_0 < m$. Suppose that $\bar{t} \to \bar{T}$. It is not incentive compatible for any low type agents to stay until $\bar{T}$, as they are better of seperating even at a lower belief. Consequently, it implies that $\alpha_{\bar{t}} \to 1$ as $\bar{t} \to \bar{T}$, implying that there exists a $\bar{t}$ such that $\pi(m, \bar{t}) = \alpha_{\bar{t}} \cdot p_{\bar{t}}^H + (1-\alpha_{\bar{t}}) \cdot p_{\bar{t}}^L$. By continuity for any given $m > q_0$ there exists a smallest $\bar{t}(m)$ such that $Q(\bar{t}(m), m) = q_0$. □

Denote by $\bar{t}(m, \Delta)$ the smallest $\bar{t}$ such that the boundary condition $(\bar{t}, m)$ corresponds to the initial prior q_0. This time $\bar{t}(m, \Delta)$, together with the boundary pooling belief $\pi(m, \bar{t}(m, \Delta))$ is the sufficient boundary condition. Denote by $(V_t^L(\Delta), V_t^H(\Delta))_{t \leq \bar{t}(m, \Delta)}$ to be the value function solutions to differential equation system (B.71) and (B.73).

Lemma B.23 (Boundary condition limit as $\Delta \to \infty$)**.** *There exists* $\bar{t}(m) = \underline{\lim}_{\Delta \to \infty} \bar{t}(m, \Delta)$*.*

Proof. By construction, we have $\bar{t} < \bar{T}$ for any Δ. This implies that any sequence $\{\bar{t}(m, \Delta_n)\}_{n=1}^{\infty}$ is bounded and, thus, possesses a lower limit. □

The following proposition establishes that existence of a limit of equilibria constructed above as $\Delta \to 0$.

Proposition B.2 (Limiting equilibrium)**.** *There exists a limit that is uniform in $t \leq \bar{t}$ as $\Delta \to \infty$ as*

$$\begin{cases} \lim_{\Delta \to \infty} \left\{ V^L(\Delta), \dot{V}^L(\Delta), V^H(\Delta), \dot{V}^H(\Delta) \right\} = \left\{ V^L, \dot{V}^L, V^H, \dot{V}^H \right\}, \\ \lim_{\Delta \to \infty} \left\{ q(\Delta), k(\Delta), \gamma(\Delta) \right\} = \left\{ q, k, \gamma \right\}. \end{cases} \tag{B.74}$$

In this limit it follows that $V_t^H > \bar{v} + V_t^L$ for a $\bar{v} > 0$ for every $t < \bar{t}(m)$.

Proof. **Limit existence.** For any fixed, arbitrarily small, $\bar{v}$, there exists a $\bar{\Delta}$ such that for every $\Delta > \bar{\Delta}$ it follows that the Jacobian of the differential equation systems (B.71) and (B.73) are non-degenerate. This implies that the solutions $(V_t^L, \dot{V}_t^L, V_t^H, \dot{V}_t^H, k_t, \gamma_t)$ are uniformly continuous in $\bar{t}(m, \Delta)$ and Δ. This implies that there exists the limit in (B.74). Holding $\bar{v}$ constant, it follows that $\lim_{\Delta \to \infty} k_t = p_t^L$ for $t < \bar{t}$ with the convergence being uniform in t.

Limit properties If $k_t = p_t^L$ it follows from the proof of Proposition B.16 that the expected value to the high type is strictly bounded away from 0 and from V_t^L. This implies that for Δ sufficiently large it follows that $V_t^H(\Delta) > V_t^L(\Delta) + \bar{v}$ for $\bar{v} \in \left[0, \frac{1}{2} \cdot \sup_t \left\{ V_t^H(\infty) - V_t^L(\infty) \right\}\right]$. □

B.6.3 Limiting Equilibrium Beliefs with Stochastic Preference Shocks

In this section we show that any well-behaved equilibrium limit as $\Delta \to \infty$ of the perturbed model converges to an equilibrium that satisfies the lowest continuation surplus refinement 1.

Definition 3. *A limiting equilibrium is a sequence of value functions $V_t^i(\Delta_n)$ and beliefs $k_t(\Delta_n)$ and $q_t(\Delta_n)$ such that*

(i) Limit: there exists a uniform limit in t of[8]

$$\begin{cases} \lim_{n \to \infty} \left\{ V^L(\Delta_n), \dot{V}^L(\Delta_n), V^H(\Delta_n), \dot{V}^H(\Delta_n) \right\} = \left\{ V^L, \dot{V}^L, V^H, \dot{V}^H \right\}, \\ \lim_{n \to \infty} \left\{ q(\Delta_n), k(\Delta_n), \gamma(\Delta_n) \right\} = \left\{ q, k, \gamma \right\}. \end{cases} \tag{B.75}$$

(ii) The limiting beliefs (q_t, k_t, γ_t), value functions $(V_t^L, V_t^H)_{t \geq 0}$, and churning strategies (e_t^L, e_t^H) are an equilibrium of a game in which $\varepsilon = 0$ and $\Delta = +\infty$.

[8] The convergence of $\gamma_t(\Delta_n) \to \gamma_t$ is equivalent to $k_t(\Delta_n)$ converging in C^1 to k_t.

As before, define $\mathbb{T} \stackrel{def}{=} \{t: V_t^L = 0\} \cup \{t: V_t^H = 0\}$.

Lemma B.24. *Suppose the belief process $(q_t, k_t)_{t\in[0,\bar{t}]}$ is right continuous. For any limiting equilibrium in which $\alpha_{\bar{t}} < 1$, i.e., such that $q_{\bar{t}} < p_{\bar{t}}^H$, it must be the case that $k_t = p_t^L$ for every $t \in \mathbb{T}$.*

Proof. Consider date t such that $V_t^H > 0$ and $V_t^L > 0$. This implies that $V_t^H(\Delta_n) > 0$ and $V_t^L(\Delta_n) > 0$ for a sufficiently high n. This implies that

$$\begin{aligned}
\lim_{n\to\infty} k_t(\Delta_n) &= \lim_{n\to\infty}\left[\frac{p_t^L \cdot (1-\alpha_t(\Delta_n)) \cdot G(V_t^L(\Delta_n)) + p_t^H \cdot \alpha_t(\Delta_n) \cdot G(V_t^H(\Delta_n))}{(1-\alpha_t(\Delta_n)) \cdot G(V_t^L(\Delta_n)) + \alpha_t(\Delta_n) \cdot G(V_t^H(\Delta_n))}\right] \\
&= \lim_{n\to\infty}\left[\frac{p_t^L \cdot (1-\alpha_t(\Delta_n)) \cdot e^{\Delta_n \cdot \left(V_t^H(\Delta_n) - V_t^L(\Delta_n)\right)} + p_t^H \cdot \alpha_t(\Delta_n)}{(1-\alpha_t(\Delta_n)) \cdot e^{\Delta_n \cdot \left(V_t^H(\Delta_n) - V_t^L(\Delta_n)\right)} + \alpha_t(\Delta_n)}\right]
\end{aligned}$$

This limit is equal to p_t^L if $V_t^L < V_t^H$ and p_t^H if $V_t^H > V_t^L$, with the latter leading to a contradiction with the equilibrium for Δ_n sufficiently high.

Suppose at time t we have $V_t^H = V_t^L > 0$. Define times t_1 and t_2 as

$$t_1 \stackrel{def}{=} \sup\{s \le t: \quad V_s^L < V_s^H\}, \qquad t_2 \stackrel{def}{=} \inf\{s \ge t: \quad V_s^L < V_s^H\}.$$

This implies that $V_t^L = V_t^H$ for $t \in [t_1, t_2]$ and, in addition, requires that $\gamma_t = 0$ for $t \in [t_1, t_2]$.

(i) Case $t_1 > 0$. In this case there exists an $\epsilon > 0$ such that $V_s^H > V_s^L$ for every $s \in (t_1 - \epsilon, t_1)$. By the above argument, it implies that $k_s = p_s^L$ and, since $\gamma_s = 0$ for $s \in [t_1, t_2]$, it implies that $k_t = p_t^L$.

(ii) Case $t_2 < \bar{t}$. In this case there exists an $\epsilon > 0$ such that $V_s^H > V_s^L$ for every $s \in (t_2, t_2 + \epsilon)$. By the above argument, it implies that $k_s = p_s^L$ and, since $\gamma_s = 0$ for $s \in [t_1, t_2]$, it implies that $k_t = p_t^L$.

(iii) Case $t_1 = 0$ and $t_2 = \bar{t}$. This implies that $\pi(k_t, -t)$ is constant for all $t \in [0, \bar{t}]$. We know that by optimality of optimal stopping it requires that $k_t < q_t$. However it implies a jump in beliefs at $\bar{t}$, which would contradict $V_{\bar{t}-}^H = V_{\bar{t}-}^L$ as the types leaving in the final instance will be better. This leads to a contradiction.

□

B.7 Candidate D1 Definition, Construction, and Possibility for Equilibrium Nonexistence

So far we have considered the lowest continuation surplus refinement 1 that we micro found in Sections B.4 and B.6 with the perturbation approach to the model. A natural question, which we discussed already in Section B.2.2 is why we do not pursue other refinements, such as Divinity. In this section we offer an extension of the definition of divinity to our dynamic two-player informed game by considering the joint expected payoff of the intermediary-agent pair. This is the focus of Section B.7.1. We formally prove in Section B.7.2 that the equilibrium we construct in the main text, as well as in Section B.5 of this Online Appendix is the unique equilibrium that may survive the divinity refinement 4.– we do so in the context of a binary model for tractability, but the logic also extends to a continuum of types. These results show that if a divine equilibrium does exist, then it coincides with the one we construct in the main text of the paper and this Online Appendix B. We show in Lemma B.30 of Section B.7.2 that such equilibrium survives the divinity refinement if the quiet period is not too long, as proxied by a sufficiently large intermediary outside option V, while there may be no divine equilibrium if the outside option V is very low and the resulting quiet period is too long. We further illustrate this logic in Section B.7.2.1 graphically in a we provide a semi-analytic example. The reason for such non-existence is that the initial quiet period offers substantial rents to lower skilled intermediary-agent pairs, which leads higher types to be more willing to deviate at the start of the game. Given the uniqueness of the candidate equilibrium, this leads to potential nonexistence of an equilibrium satisfying the divinity definition 4 for some parameter values.

B.7.1 Definition of Divinity in our Setting

Signaling games have a special feature that the agent only acts once. Dynamic signaling is not that the agent cannot act sooner, but it is that the buyers can make interim offers. As long as clients are dispersed, an individual client cannot make such a deviation.

Lemma B.25. *The set of Perfect Bayesian Equilibria of the dynamic game coincides with that of a static game in which the intermediary and the agent jointly commit at $t = 0$ to contingent separation date $\tau(p)$, but this time is not observed by clients until it occurs.*[9]

The only distinction between a standard signaling model and this signaling model is the belief process Q. This is not present in the classic models in which the flow value prior to signaling is independent of beliefs. However the claim is that if it were present then the result would be the same.

Remark 1. *If a Perfect Bayesian Equilibrium can be supported in the initial dynamic game, then it can be supported in a strategic form game. Divinity is not defined for dynamic games, but it is well defined for the static game in which at $t = 0$ the intermediary and the agent commit to a separation time.*

The ex-ante set of types is $[\underline{p}, \overline{p}]$. Define by $(q_t, k_t)_{t\geq 0}$ to be the equilibrium belief processes. Define $W_t(p_t)$ to be the on-path expected continuation value to the intermediary-agent pair at time t given ex-ante type

[9] Every equilibrium of the dynamic game is an equilibrium of the strategic-form game.

p along the path of good performance

$$W_t(p) \overset{def}{=} \sup_{\tau} \mathrm{E}_p \left[\int_t^{\tau} e^{-r(s-t)} \cdot \Big(A(q_s) - rV \Big) \, ds + e^{-r(\tau - t)} \cdot U\big(\pi(p,\tau), k_\tau\big) \mid X_t = t \right] + V. \tag{B.76}$$

Define $V(p,t)$ as the expected ex-ante value of stopping at time t, i.e.,

$$\begin{aligned} W(p,t) &\overset{def}{=} \mathrm{E}_p \left[\int_0^{t\wedge\eta} e^{-rs} \cdot \Big(A(q_s) - rV \Big) \, ds + e^{-rt\wedge\eta} \cdot U\big(p_{t\wedge\eta}, k_{t\wedge\eta}\big) \right] + V \qquad \text{(B.77)} \\ &= \int_0^t e^{-rs} \cdot \Big(p + (1-p)\cdot e^{-\lambda s}\Big) \cdot \Big(A(q_s) - rV \Big) \, ds + e^{-rt} \cdot \Big(p + (1-p)\cdot e^{-\lambda t}\Big) \cdot U\Big(\pi(p,t), k_t\Big) + V. \end{aligned}$$

Suppose the intermediary and the agent of ex-ante type p separate and the agent is perceived as having a private ex-ante type l by clients. The joint ex-ante surplus of the agent and intermediary is

$$\begin{aligned} W(p,t,l) \overset{def}{=} &\int_0^t e^{-rs} \cdot \Big(p + (1-p)\cdot e^{-\lambda s}\Big) \cdot \Big(A(q_s) - rV \Big) \, ds \\ &+ e^{-rt} \cdot \Big(p + (1-p)\cdot e^{-\lambda t}\Big) \cdot U\Big(\pi(p,t), \pi(l,t)\Big) + V. \end{aligned} \tag{B.78}$$

It is valuable for the intermediary and the agent to separate at time t only if the if the client assigns the agent an ex-ante belief l such that $W_0(p) < W(p,t,l)$.

(i) Denote by $D^0(p|t)$ the set of ex-ante beliefs l for which the agent is indifferent between staying on path or choosing action t:

$$D^0(p|t) \overset{def}{=} \{l: \quad W_0(p) = W(p,t,l)\} = \{d_t(p)\}, \tag{B.79}$$

where $d_t(p)$ is the unique solution to

$$W_0(p) = V(p,t,d_t(p)), \tag{B.80}$$

which is well-defined as $W(p,t,l)$ is strictly increasing in l.

(ii) Denote by $D(p|t)$ the set of ex-ante beliefs l for which it is preferable for the intermediary-agent pair to break their relationship at time t given good performance:

$$D(p|t) \overset{def}{=} \{l: \quad W_0(p) < W(p,t,l)\} = \{l > d_t(p)\}. \tag{B.81}$$

Definition 4. *Consider three types of refinements.*

(i) ***Divinity (D1)*** *criterion eliminates off-path action t by type p if there exists a type p' such that*

$D(p|t) \cup D^0(p|t) \subset D(p'|t)$.

(ii) ***Universal Divinity*** *criterion eliminates off-path action t by type p if* $D(p|t) \cup D^0(p|t) \subset \cup_{p' \neq p} D(p'|t)$.

(iii) ***The Never Weak Best Response (NWBR)*** *criterion eliminates off-path action t by type p if* $D^0(p|t) \subset \cup_{p' \neq p} D(p'|t)$.

The following result applies Cho and Sobel (1990) to payoff structure specified by (B.3).

Lemma B.26. *In the model considered in Section 2 and summarized by the payoff structure* (B.78)*, D1 is equivalent to both Universal Divinity and NWBR.*

Proof. Suppose action t by type p is eliminated by D1, i.e., there exists a type p' such that

$$D^0(p|t) \overset{(i)}{\subseteq} D(p|t) \cup D^0(p|t) \subset D(p'|t) \overset{(i)}{\subseteq} \bigcup_{\hat{p} \neq p} D(\hat{p}|t). \tag{B.82}$$

Inclusion (ii) in (B.82) implies that type p is eliminated by Universal Divinity. Inclusions (B.82) (i) and (ii) imply that it is also eliminated by NWBR. The monotonicity of the preferences in beliefs, it follows that $D(p|t) = (d_t(p), 1]$ and $D^0(p|t) = \{d_t(p)\}$. The D1 criterion eliminates type p if there exists a p' such that $d_t(p) > d_t(p')$. Moreover, by choosing $p' = \arg\min d_t(p)$, D1 is equivalent to Universal Divinity. NWBR eliminates type p if $d_t(p) > \min_{p'} d_t(p')$, which is equivalent to D1. □

Following Lemma B.26 we refer to the D1 refinement keeping in mind its equivalence to both Universal Divinity and NWBR.

Corollary B.3 (D1 off-path beliefs)**.** *The equilibrium survives the D1 criterion if*

$$k_t \in \pi\Big(\arg\min_p d_t(p), t\Big) = \arg\min_p \pi(d_t(p), t) \qquad \textit{for every} \qquad t \notin \mathbb{T}. \tag{B.83}$$

Relationship to a Single Informed Agent Model as Motivation for D1

Our model features two informed players (the agent and the intermediary) who are interested in maximizing overall surplus, but compete on wages. Our definition of the D1 belief refinement in 4 applies on the level of agent and intermediary, meaning that they agree on a deviation jointly, and then split the benefits/costs via wages. Natural questions may arise in why this is a natural interpretation of the D1 refinement, which is normally applied to a single-informed agent games, to our setting. In what follows we show that we can formulate a single informed player model in which the payoff of this player coincides with the joint payoff of the intermediary-agent pair. The definition of D1 in our setting 4 will then coincide with the classic definition of D1 following Cho and Sobel (1990). Our subsequent analysis then shows the nonexistence of the D1 equilibrium for some parameter values, highlighting the value to our novel approach of equilibrium selection via perturbation methods in Sections B.2.

Consider a single-agent model in which a single player of ex-ante skill $\theta \in \{0, 1\}$ operates a firm. The firm's revenue are a function $A(\cdot)$ of the expected ability of the player. The player of ability $\theta = 1$ always generates good performance, while the player of ability $\theta = 0$ generates a loss with intensity λ. The player has a binary signal about his ability, which we can identify with his private prior $\tilde{p} \in \{p^L, p^H\}$ about θ. The player has a profitable investment project that he can undertake at no cost, while continuing to operate his existing firm. The project yields expected value V that is constant and independent of the player's ability.

The expected value to the player from undertaking the project at time t depends on the beliefs of clients about which types undertake the project at that time. Denote these beliefs by k_t, while denote by q_t the belief about the agents who have not undertaken this new project at time t. The expected value for the player from undertaking the project at time t and being perceived as ex-ante type l is given by $W(p, t, l)$ defined in (B.78). This single player game is monotone in client beliefs, meaning that the three belief refinements outlined in 4 coincide with the classic notion of D1 refinement outlined in Cho and Sobel (1990). Everything we do from here onwards can be applied to this single informed-player game. The key distinction from the games considered in Cho and Sobel (1990) is the dependence of the payoff of the agent on client beliefs q_t prior to undertaking the irreversible public action. This both generates a quiet period for both players, but also limits the applicability of using the classic D1 belief refinement as we show below.

B.7.2 Divine Equilibrium Construction and Conditions for Existence

In this section we construct the unique candidate equilibrium that survives the D1 refinement as defined by 4. Lemma B.29 shows that it must be separating and feature a quiet period $[0, t_1^*]$ during which all types are retained as long as they perform well, a churning period $[t_1^*, t_2^*]$ during which low type p^L agents are gradually let go, and a final time $\bar{t}$ when the high skilled p^H agents are let go. We show in Lemma B.30 that this equilibrium survives the divinity refinement 4 if the quiet period is not too long, as proxied by a sufficiently high intermediary outside option V. At the same time, surprisingly, if V is very low, then the initial quiet period is so long that no divine equilibrium may exist.

As in Section B.5 suppose the types are binary and characterized by $p \in \{p^L, p^H\}$. Denote by $\mathbb{T}^I(p^L)$ and $\mathbb{T}^I(p^H)$ to be the set of times during which types L and H are willing to separate. I.e., $\mathbb{T}^I(p^L)$ and $\mathbb{T}^I(p^H)$ are indifference sets for both the high and the low types.

Lemma B.27 (Final period separation)**.** *Suppose the equilibrium satisfies 4. Then there exists a* $\bar{t} = \sup \mathbb{T}^I(p^H) \geq \sup \mathbb{T}^I(p^L)$*. Moreover, it must be the case that* $l_{\bar{t}} = p^H$ *and* $\bar{t} \in \mathbb{T}^I(p^L)$*.*

Proof. The fact that separation times are bounded from above follows from Lemma A.17. Moreover, it can't be the case that $\sup \mathbb{T}^I(p^L) > \sup \mathbb{T}^I(p^H)$ since a low type agent would prefer to separate at $\sup \mathbb{T}^H$ together with the high type, rather than postpone separation and still be identified as a low type in equilibrium.

Suppose there is pooling at $\bar{t}$. Then consider an ε deviation to separate at $\bar{t}+\varepsilon$. Consider the indifference beliefs $d_{\bar{t}}(p^i)$ and $d_{\bar{t}}(p^H)$ that clients assign to agent-intermediary pairs that do not separate during $t \in [\bar{t}, \bar{t}+\varepsilon]$. These indifference beliefs must satisfy

$$U\Big(\pi(p^i,\bar{t}), \pi(l_{\bar{t}},\bar{t})\Big) = \int_0^\varepsilon e^{-rt} \cdot \Big(\pi(p^i,\bar{t}) + (1-\pi(p^i,\bar{t})) \cdot e^{-\lambda t}\Big) \cdot \Big[A\Big(\pi(d_{\bar{t}}(p^i),t)\Big) - rV\Big]\, dt$$
$$+ e^{-r\varepsilon}\pi(p^i,\bar{t}) \cdot u_1\Big(\pi(d_{\bar{t}}(p^i),\bar{t})\Big) + e^{-(r+\lambda)\varepsilon}\Big(1-\pi(p^i,\bar{t})\Big) \cdot u_0\Big(\pi(d_{\bar{t}}(p^i),\bar{t})\Big).$$

It must be the case that $d_{\bar{t}}(p^i) \geq l_{\bar{t}}$ since the agent and the intermediary sacrifices rV during the period $[0,\varepsilon]$. For ε sufficiently small it must be the case that $d_{\bar{t}}(p^i)$ is close to $l_{\bar{t}}$ such that

$$A\Big(\pi(d_{\bar{t}}(p^i),t)\Big) - rV < A\Big(\pi(l_{\bar{t}},t)\Big)$$

for all $t \in [0,\varepsilon]$. Consider the cash flow process $C = (C_t)_{t\geq 0}$ defined as $C_t = A(\pi(l_{\bar{t}},t))$ and cash flow process $\hat{C} = (\hat{C}_t)_{t\geq 0}$ defined as

$$\hat{C}_t = \begin{cases} A\Big(\pi(d_{\bar{t}}(p^H),t)\Big) - rV & \textit{if} \quad t \in [\bar{t}, \bar{t}+\varepsilon), \\ A\Big(\pi(d_{\bar{t}}(p^H),t)\Big) & \textit{if} \quad t \geq \bar{t}+\varepsilon. \end{cases}$$

By definition of $d_{\bar{t}}(p^H)$ the time $\bar{t}$ continuation value from the cash flows C and $\hat{C}$ is identical for type $\tilde{p}_0 = p^H$. However $C_t > \hat{C}_t$ if and only if $t \in [\bar{t}, \bar{t}+\varepsilon]$ cash flows C and $\hat{C}$ satisfy the single-crossing condition of Lemma A.1. This implies that the time $\bar{t}$ continuation value for type $p^L < p^H$ agent is strictly higher for cash flow stream C than cash flow stream $\hat{C}$. This, in turn, implies that the indifference belief $d_{\bar{t}}(p^L)$ must be strictly higher than $d_{\bar{t}}(p^H)$. This implies that if clients see an agent stay beyond time $\bar{t}$, following the D1 criterion, clients should assign the high type beliefs $l_t = p^H$ for $t > \bar{t}$. This then implies that if $\bar{t} \in \mathbb{T}^I(p^H)$, then it requires that $l_{\bar{t}} = p^H$, implying that there is no pooling at time $\bar{t}$.

Define $\bar{t}_L = \sup \mathbb{T}^I(p^L)$ and suppose that $\bar{t}_L < \bar{t}$. This implies that to wait until $\bar{t}$ the lowest type requires $d_{\bar{t}}(p^L) > p^H$. By continuity, it follows that if the the agent-intermediary pair choose to separate at $\bar{t}-\varepsilon$, then for a sufficiently low ε it follows that $d_{\bar{t}-\varepsilon}(p^L) > d_{\bar{t}-\varepsilon}(p^H)$. Following the D1 refinement it then must be the case that $l_{\bar{t}-\varepsilon} = p^H$, implying that type p^H agent would strictly prefer separating at $\bar{t}-\varepsilon$ at the correct belief of p^H, rather than wait until time $\bar{t}$. $\square$

Lemma B.28 (Divine equilibrium must be separating)**.** *Suppose $A(x) = x$, $L = 0$, and $\underline{p} \geq 1/2$. Then there cannot be pooling prior to $\bar{t}$ and the high type only separates in equilibrium at time $\hat{t}$.*

Proof. Define $\hat{t} \stackrel{def}{=} \sup\left\{\mathbb{T}^I(p^H) \cap [0, \bar{t})\right\}$. Suppose $\hat{t} = \bar{t}$. This implies that there exists a sequence of times $\{t_n\} \subset \mathbb{T}^I(p^H)$ such that $t_n \to \bar{t}$. It must be the case that $\{t_n\} \subset \mathbb{T}^I(p^L)$ as well since, otherwise, it would not be incentive compatible for the p^H type to wait until $\bar{t}$. Consider the indifference belief $d_t(p^L)$ and $d_t(p^H)$ at which types p^L and p^H are willing to separate. From the previous Lemma it follows that $d_{\bar{t}}(p^L) = d_{\bar{t}}(p^H) = p^H$. It further follows that

$$\frac{d}{dt} d_t(p^i) = \frac{rV + A(\pi(d_t(p^i))) - A(q_t)}{\partial_1 \pi(d_t(p^i), t) \cdot \partial_2 U(\pi(p^i, t), \pi(d_t(p^i), t))}.$$

For t close to $\bar{t}$ it follows from above that $d_t(p^L) < d_t(p^H)$. Hence it must be the case that $d_{t_n}(p^L) < d_{t_n}(p^H)$ for n sufficiently large, which contradicts with the existence of $\{t_n\} \in \mathbb{T}^I(p^L) \cap \mathbb{T}^I(p^H)$.

The above shows that $\hat{t} < \bar{t}$. It must still be the case that $\hat{t} \in \mathbb{T}^I(p^L) \cap \mathbb{T}^I(p^H)$. This implies that both types p^L and p^H are indifferent between separating at times $\hat{t}$ and $\bar{t}$. Then

$$\begin{cases} u_1(\pi(l_{\hat{t}}, \hat{t})) = \displaystyle\int_{\hat{t}}^{\bar{t}} e^{-r(s-\hat{t})} \cdot \Big(A(q_s) - rV - A(\pi(p^H, s))\Big)\, ds + u_1(\pi(p^H, \hat{t})), \\ u_0(\pi(l_{\hat{t}}, \hat{t})) = \displaystyle\int_{\hat{t}}^{\bar{t}} e^{-(r+\lambda)(s-\hat{t})} \cdot \Big(A(q_s) - rV - A(\pi(p^H, s))\Big)\, ds + u_0(\pi(p^H, \hat{t})). \end{cases} \tag{B.84}$$

Suppose that $l_{\hat{t}} > p^L$ meaning that there is pooling at time $\hat{t}$. Given this pooling outcome at $\hat{t}$, there exists an $\varepsilon > 0$ such that there are no separations along the equilibrium path during $(\hat{t} - \varepsilon, \hat{t} + \varepsilon)$. It must be the case that $A\left(q_{\hat{t}+}\right) - rV - A\left(\pi(l_{\hat{t}}, \hat{t})\right) \geq 0$ since there would otherwise be a contradiction with D1 with separations at $\hat{t} + \varepsilon$. Due to the linearity of $A(\cdot)$, the expected value of the average agent that leaves the intermediary at time $\hat{t}$ is then given by

$$\frac{\pi\left(l_{\hat{t}}, \hat{t}\right)}{r} \leq \frac{A\left(q_{\hat{t}+}\right) - rV}{r}.$$

The equilibrium value of the expected intermediary-agent pair is given by

$$\begin{aligned} \mathrm{E}\left[\int_{\hat{t}}^{\tau\wedge\eta} e^{-r(t-\hat{t})} \cdot (A(q_t) - rV)\, dt + \int_{\tau\wedge\eta}^{\eta} e^{-r(t-\hat{t})} A(p_t)\, dt\right] &= \mathrm{E}\left[-\int_{\hat{t}}^{\tau\wedge\eta} rV\, dt + \int_{\hat{t}}^{\eta} e^{-r(t-\hat{t})} p_t\, dt\right] \\ &> -\int_{\hat{t}}^{\infty} rV\, dt + \int_{\hat{t}}^{\infty} e^{-r(t-\hat{t})} \mathrm{E}\left[p_t\right]\, dt \end{aligned}$$

$$= \frac{q_{\hat{t}} - rV}{r}.$$

This implies that the expected value of the average intermediary-agent pair always strictly exceeds $\frac{q_t - rV}{r}$ since $A(\cdot)$ is linear. This implies that it is not incentive compatible for either type L or type H agent to separate at time $\hat{t}$, contradicting the possibility of pooling. Finally, it cannot be the case that the high type separates before all low types separate as the low type's payoff is dominated by the expected value of being perceived as the high type. □

Lemmas B.27 and B.28 above state that the low types separate prior to $\bar{t}$ and the high type separates at $\bar{t}$. Moreover, the low type agent also finds it weakly optimal to wait until time $\hat{t}$ but separates before then with certainty.

Define by t_1^* the candidate first time when it is locally optimal for the low type to leave the intermediary:

$$t_1^* \stackrel{def}{=} \inf\left\{t: \quad \pi(q_0, t) - \pi\left(p^L, t\right) - rV = 0\right\}. \tag{B.85}$$

Define by t_2^* the candidate last time when it is locally optimal for the low type to leave the intermediary:

$$t_2^* \stackrel{def}{=} \inf\left\{t: \quad 1 - \pi\left(p^L, t\right) - rV = 0\right\}. \tag{B.86}$$

Lemma B.29 (Necessary equilibrium structure)**.** *Suppose $A(x) = x$ is linear, $L = 0$, and $\underline{p} \geq 1/2$. Then in any equilibrium satisfying the D1 refinement 4 it must be the case that $\mathbb{T}^L = \{t_1^*, t_2^*\} \cup \{\bar{t}\}$ and all the low type agents separate from the intermediary gradually during $[t_1^*, t_2^*]$.*

Proof. Following Lemma B.28, the Perfect Bayesian Equilibrium satisfying the D1 refinement must be separating in types. Taken together with Lemma B.27, it follows that all low types must separate prior to $\bar{t}$, while still being indifferent to waiting until $\bar{t}$.

Consider the first time that the low type agent separates from the intermediary. Since in equilibrium it must happen at the separating belief, the intermediary-agent pair solves a stopping problem

$$\max_T \left\{ \int_0^T e^{-rt}\left(p^L + (1-p^L)e^{-\lambda t}\right)\left(A(\pi(q_0, t)) - rV\right) dt + \int_T^\infty e^{-rt}\left(p^L + (1-p^L)e^{-\lambda t}\right) A(\pi(p^L, t))\, dt \right\}.$$

Under the stated assumptions, the revenue function $A(\pi(p,t))$ is concave in t, implying that the there exists a unique t_1^*, defined by (B.85) when it is optimal to let go of the low type agent.

Similarly, consider the last time when it is optimal for the low type to leave the intermediary. Since, following Lemma B.27 all low type agents must leave prior to $\bar{t}$, it must be the case that at this date $\hat{t}$

the remaining types are only the high types, meaning that $q_{\hat{t}} = 1$. Local optimality then implies that this date $\hat{t}$ must coincide with t_2^* defined in (B.86).

Finally, since $A(x) \equiv x$, it follows that $A(\pi(p,t))$ is concave in t for $p \geq 1/2$ and $t \geq 0$. This implies that there cannot be a positive gap in separations during $[t_1^*, t_2^*]$. The reason is that if there ever was a gap $(\hat{t}_1, \hat{t}_2)$, then local optimality at $\hat{t}_1$ implies that $A(q_{\hat{t}_1}) - A(\pi(p^L, \hat{t}_1)) = rV$. The concavity property of $A(\pi(p,t))$ then would imply that $A(\pi(q_{\hat{t}_1}, t - \hat{t}_1)) - A(\pi(p^L, t)) < rV$ for $p \in (\hat{t}_1, \hat{t}_2]$, implying a contradiction with the optimality of waiting to separate at $\hat{t}_2$ for the low type agent. □

The above lemmas show that in order for the Perfect Bayesian Equilibrium to satisfy the D1 refinement, it must be the case that the high type separates at time $\bar{t}$, the low types separate gradually during $[t_1^*, t_2^*]$, and the low type finds it weakly optimal to wait until $\bar{t}$. In order for this to be an equilibrium, it must be the case that the beliefs disciplined by D1 during the quiet period $[0, t_1^*)$ are consistent with the above strategies.

For the equilibrium to satisfy the D1 refinement, it must be the case that for $t \in [0, t_1^*) \cup (t_2^*, \bar{t}) = [0, \bar{t}] \setminus \mathbb{T}^L$ the low type is weakly more likely to deviate than the high type, meaning that $d_t(p^L) \leq d_t(p^H)$. If this were not at satisfied at any point $\hat{t}$, then the D1 refinement would specify a high type off-equilibrium belief $k_{\hat{t}} = p^H_{\hat{t}}$, which would break the equilibrium conjectured above by making all types willing to separate at $\hat{t}$.

Lemma B.30 (Existence and nonexistence of a Divine Equilibrium)**.** *Suppose $A(\pi(p,t))$ is weakly concave in t. Then there exist two thresholds $\underline{V} < \overline{V}$. If the intermediary's outside option $V > \overline{V}$ then there exists a unique divine equilibrium characterized in Lemma B.29 and the clients' belief $k_t = p^L_t$ for all $t < \bar{t}$. If the intermediary's outside option $V < \underline{V}$, however, then a divine equilibrium does not exist.*

Proof. Lemma B.29 specifies the unique separation dynamics that may take place in a divine equilibrium. We need only verify if such separation dynamics are consistent with off path beliefs specified by the D1 refinement in (B.83).

Case $V > \overline{V}$. In the equilibrium constructed in Lemma B.29 the low type is indifferent between separating at time t_1^* and waiting until pooling with type $\bar{t}$. At the same time, the high type finds it strictly optimal to wait until separation at time $\bar{t}$. Consider $V = A(q_0) - A(p^L_0)$. This implies that the game begins immediately with the churning period. At the start of this churning period, by construction, $d^L_0 = p^L_0$, while $d^H_0 = p^H_0$. By continuity of the solution to (B.80), this implies that there exists $\overline{V} < A(q_0) - A(p^L_0)$ such that there is a positive length quiet period, i.e., $t_1^* > 0$ and $d^L_t < d^H_t$ for $t < t_1^*$.

Case $V < \underline{V}$. Index the churning period $[t_1^*(V), t_2^*(V)]$, the average belief process $q_t(V)$, and the indifference beliefs $d_t(p^L; V)$ and $d_t(p^H; V)$ solving (B.80) by the intermediary's outside option V. At $V = 0$

the intermediary equilibrium features no churning and, consequently, $d_t(p^L;0) = d_t(p^H;0) = \pi(q_0,t)$. In what follows, we show that $\frac{d}{dV}d_t(p^H;V)\big|_{V=0} < \frac{d}{dV}d_t(p^L;V)\big|_{V=0}$, implying that the equilibrium constructed in Lemma B.29 does not satisfy the divinity refinement at the start of the quiet period when V is sufficiently low.

During the quiet period $t \in [t_1^*(V), t_2^*(V)]$ it follows that

$$A(q_t(V)) - A(p_t^L) = rV. \tag{B.87}$$

The indifference condition to separate at time $\bar{t}(V)$ is given by

$$\begin{aligned} &\int_0^{t_1^*(V)} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A(q_s(V)) - rV - A(p_s^L)\right)\, ds + U(p^L,p^L) \\ =&\int_0^{\bar{t}(V)} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A(q_s(V)) - rV - A(p_s^H)\right)\, ds + U(p^L,p^H). \end{aligned}$$

Given the indifference condition (B.87), the derivative with respect to V of both the left and right hand side is given by

$$\begin{aligned} &\int_0^{t_1^*(V)} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A'(q_s(V))\cdot q_s'(V) - r\right)\, ds \\ =&\bar{t}'(V)\cdot \left(p^L + (1-p^L)\cdot e^{-\lambda \bar{t}(V)}\right) \cdot \left(A(q_{\bar{t}(V)}(V)) - rV - A(p^H_{\bar{t}(V)})\right) \\ &+\int_0^{\bar{t}(V)} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A'(q_s(V))\cdot q_s'(V) - r\right)\, ds \end{aligned} \tag{B.88}$$

Note that $q_{\bar{t}(V)} = p^H_{\bar{t}(V)}$. Consequently (B.88) can be simplified to

$$\begin{aligned} &\int_0^{t_1^*(V)} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A'(q_s(V))\cdot q_s'(V) - r\right)\, ds \\ =& -\bar{t}'(V)\cdot \left(p^L + (1-p^L)\cdot e^{-\lambda \bar{t}(V)}\right) \cdot rV \\ &+\int_0^{\bar{t}(V)} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A'(q_s(V))\cdot q_s'(V) - r\right)\, ds \end{aligned} \tag{B.89}$$

Consider now $V=0$. In this case $t_1^*(V) = +\infty$, $q_s(V) = \pi(q_0,s)$, and $q_s'(0) = 0$. Then (B.89) becomes

$$\begin{aligned} &\int_0^{\infty} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A(\pi(q_0,s))\cdot 0 - r\right)\, ds \\ =& -\bar{t}'(0)\cdot \left(p^L + (1-p^L)\cdot e^{-\lambda \bar{t}(0)}\right) \cdot r \\ &+\int_0^{\infty} e^{-rs} \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) \cdot \left(A'(\pi(q_0,s))\cdot 0 - r\right)\, ds \end{aligned}$$

The above indifference condition implies that

$$\lim_{V\to 0} \bar{t}'(V) \cdot \left(p^L + (1-p^L)\cdot e^{-\lambda \bar{t}(V)}\right) \cdot rV = 0. \tag{B.90}$$

Consider now the indifference beliefs $d_0(p^L;V)$ and $d_0(p^H;V)$ for the low and high type respectively. The indifference belief $d_0^i(V)$ solves

$$U(p^i, d_0^i(V)) = \int_0^{\bar{t}(V)} e^{-rs} \cdot \left(p^i + (1-p^i)\cdot e^{-\lambda s}\right) \cdot \left(A(q_s(V)) - rV - A(p_s^H)\right)\, ds + U(p^i, p^H).$$

Differentiating both sides with respect to V obtain

$$\begin{aligned}\partial_2 U(p^i, d_0^i(V)) \cdot \frac{d}{dV} d_0^i(V) = &\int_0^{\bar{t}(V)} e^{-rs} \cdot \left(p^i + (1-p^i)\cdot e^{-\lambda s}\right) \cdot \left(A(q_s(V)) \cdot q_s'(V) - r\right)\, ds \\ &-\bar{t}'(V) \cdot e^{-r\bar{t}(V)} \cdot \left(p^i + (1-p^i)\cdot e^{-\lambda \bar{t}(V)}\right) \cdot rV.\end{aligned}$$

It follows from (B.90) that at $V = 0$ obtain

$$\partial_2 U(p^i, q_0) \cdot \frac{d}{dV} d_0^i(0) = -\int_0^{\infty} e^{-rs} \cdot \left(p^i + (1-p^i)\cdot e^{-\lambda s}\right) \cdot r\, ds$$

where we used the fact that $d_0^i(0) = q_0$. For the divine equilibrium to not exist we need $\frac{d}{dV} d_0^H(0) < \frac{d}{dV} d_0^L(0)$, which can be expressed as

$$\begin{aligned}
\frac{-\int_0^\infty e^{-rs}\cdot\left(p^H + (1-p^H)\cdot e^{-\lambda s}\right)\cdot r\, ds}{\partial_2 U(p^H, q_0)} &< \frac{-\int_0^\infty e^{-rs}\cdot\left(p^L + (1-p^L)\cdot e^{-\lambda s}\right)\cdot r\, ds}{\partial_2 U(p^L, q_0)} \\
\frac{\int_0^\infty e^{-rs}\cdot\left(p^H + (1-p^H)\cdot e^{-\lambda s}\right) ds}{\partial_2 U(p^H, q_0)} &> \frac{\int_0^\infty e^{-rs}\cdot\left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) ds}{\partial_2 U(p^L, q_0)} \\
\partial_2 U(p^L, q_0) \cdot \int_0^\infty e^{-rs}\cdot\left(p^H + (1-p^H)\cdot e^{-\lambda s}\right) ds &> \partial_2 U(p^H, q_0) \cdot \int_0^\infty e^{-rs}\cdot\left(p^L + (1-p^L)\cdot e^{-\lambda s}\right) ds \\
\partial_2 U(p^L, q_0) \cdot \left(\frac{p^H}{r} + \frac{1-p^H}{r+\lambda}\right) &> \partial_2 U(p^H, q_0) \cdot \left(\frac{p^L}{r} + \frac{1-p^L}{r+\lambda}\right) \\
\left(p^L \cdot u_1'(q_0) + (1-p^L)\cdot u_0'(q_0)\right) \cdot \left(\frac{p^H}{r} + \frac{1-p^H}{r+\lambda}\right) &> \left(p^H \cdot u_1'(q_0) + (1-p^H)\cdot u_0'(q_0)\right) \cdot \left(\frac{p^L}{r} + \frac{1-p^L}{r+\lambda}\right) \\
\left(p^L\left(\frac{p^H}{r} + \frac{1-p^H}{r+\lambda}\right) - p^H\left(\frac{p^L}{r} + \frac{1-p^L}{r+\lambda}\right)\right) u_1'(q_0) &< \left((1-p^H)\left(\frac{p^L}{r} + \frac{1-p^L}{r+\lambda}\right) - (1-p^L)\left(\frac{p^H}{r} + \frac{1-p^H}{r+\lambda}\right)\right) u_0'(q_0) \\
\frac{p^L - p^H}{r+\lambda} \cdot u_1'(q_0) &> \frac{p^L - p^H}{r} \cdot u_0'(q_0) \\
r \cdot u_1'(q_0) &< (r+\lambda) \cdot u_0'(q_0).
\end{aligned} \tag{B.91}$$

Note that

$$
\begin{aligned}
r \cdot u_1'(q_0) &= r \cdot \int_0^\infty e^{-rt} \cdot A'(\pi(q_0,t)) \cdot \frac{e^{-\lambda t}}{(q_0 + (1-q_0)\cdot e^{-\lambda t})}\, dt \\
&= r \cdot \int_0^\infty e^{-rt} \cdot \frac{1}{q_0(1-q_0)} \cdot \frac{d}{dt} A(\pi(q_0,t))\, dt \\
&= -\frac{1}{q_0(1-q_0)} \cdot \int_0^\infty \frac{d}{dt} A(\pi(q_0,t))\, de^{-rt} \\
&= \frac{1}{q_0(1-q_0)} \cdot \frac{d}{dt} A(\pi(q_0,t))\Big|_{t=0} + \frac{1}{q_0(1-q_0)} \cdot \int_0^\infty e^{-rt} \underbrace{\frac{d^2}{dt^2} A(\pi(q_0,t))}_{\leq 0}\, dt \\
&< \frac{1}{q_0(1-q_0)} \cdot \frac{d}{dt} A(\pi(q_0,t))\Big|_{t=0} + \frac{1}{q_0(1-q_0)} \cdot \int_0^\infty e^{-(r+\lambda)t} \underbrace{\frac{d^2}{dt^2} A(\pi(q_0,t))}_{\leq 0}\, dt = (r+\lambda) \cdot u_0'(q_0).
\end{aligned}
$$

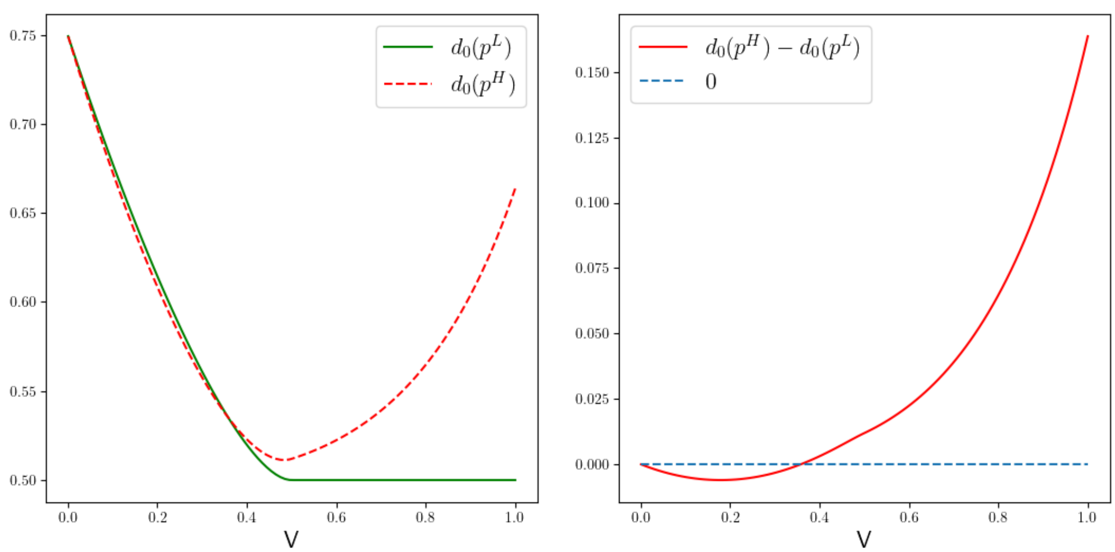
The above chain of inequalities relies on the fact that $A(\pi(p,t))$ is weakly concave in t and $e^{-rt} > e^{-(r+\lambda)t}$. Consequently, inequality (B.91) is satisfied whenever $A(\pi(p,t))$ is weakly concave in t, which then implies that $\frac{d}{dV} d_0^H(0) < \frac{d}{dV} d_0^L(0)$. Since $d_0^L(0) = d_0^H(0) = q_0$, it then implies, by continuity, that there exists an $\underline{V}$ such that for all $V < \underline{V}$ the equilibrium constructed in Lemma B.29 does not satisfy the divinity refinement (B.83) during the initial quiet period. □

Lemma B.30 shows analytically that when a divine equilibrium exists, it takes the quiet-churning structure as the equilibrium we construct in Section 3 of the main text. We show, however, that when the intermediary's outside option V is very low, then the resulting quiet period delivers sufficient rents to low skilled agents that they become less likely to deviate. The greater propensity of higher types to separate, as captured by $d_0^H < d_0^L$ for low $V's$ occurs due to the fact that if the initial quiet period is very long, then the unique candidate divine equilibrium constructed in Section B.7.2 looks very similar to pooling from the perspective of the high type. Put differently, the churning dynamics occur sufficiently late in the game that the equilibrium rents obtained by the low type intermediary-agent pair exceed that of the high type intermediary-agent pair, leading the latter to be more likely to deviate. This, however, implies that a divine equilibrium may not exist since if clients attribute an off-path deviation to a high type at the start of the game, then all types would separate, resulting in immediate separation for all types. Such pooling, however is again precluded by the divinity refinement as the high type intermediary-agent pair would be more willing to delay separation. This implies that if the initial quiet period is very long, which holds if, for example, the intermediary's outside option V is very low, then there may not be a Perfect Bayesian Equilibrium satisfying the divinity refinement 4. In the following Section we provide a semi-analytic example of the ranking of indifference beliefs $d_0(p^L;V)$ and $d_0(p^H;V)$ as a function of the intermediary's outside option.

B.7.2.1 Numerical Illustration of Divine Equilibrium Existence and Nonexistence

Figure 1 provides an numerical illustration of this argument by plotting the indifference beliefs $d_0(p^L; V)$ and $d_0(p^H; V)$ at the start of the game of the low and high type agents as a a function of the intermediary's outside option V. We obtain Figure 1 in a semi-analytic parametrization of the model that we derive below in order to maximize tractability and precision of the numerical exercise.

Figure 1a plots the indifference beliefs $d_0(p^L)$ and $d_0(p^H)$, whereas Figure 1b plots the difference $d_0(p^H) - d_0(p^L)$ as a function of V. We see in Figure 1b that there exists an $\underline{V} = \overline{V}$ such that $d_0(p^H; V) < d_0(p^L; V)$ for $V < \underline{V}$ and $d_0(p^H; V) > d_0(p^L; V)$ for $V > \underline{V}$. For the equilibrium to survive the divinity refinement 4 it must then be the case that clients attribute separations at $t = 0$ to the high type agent, meaning that $l_0 = p^H$. This, however, violates the incentive compatibility of the unique candidate divine equilibrium constructed in Section B.7.2 since all intermediary-agent pairs would then prefer to separate immediately at $t = 0$. Following the arguments of Section B.7.2, however, such pooling at $t = 0$ also violates the divinity refinement 4 as it then creates an incentive for higher skilled agents to delay separation to signal their ability.



(a) Indifference beliefs $d_0(p^L)$ and $d_0(p^H)$ at $t = 0$ as a function of the intermediary's outside option V.

(b) The difference in indifference beliefs. The equilibrium does not survive the divinity criterion whenever $d_0(p^H) - d_0(p^L) < 0$.

Figure 1: Indifference beliefs and equilibrium nonexistence for low intermediary values V. Model parameters: $r = \lambda = 0.5$, $p^L = 0.5$, $p^H = 1$, $q_0 = 0.75$, $A(x) \equiv x$.

The intuition is that for a low $V < \underline{V}$, the quiet period is very long. Such long period of pooling is beneficial for the low types and makes them less willing to deviate at $t = 0$ relative to the high types.

The intuition for why D1 breaks down this initial long pooling period is identical to why D1 breaks pooling in the final period – the high type is willing to sacrifice some short-term rates to avoid prolonged pooling with the low type, regardless of whether it takes place early or later in the game. Consequently, even though the divinity criterion disciplines the game starting from the churning period, as we have shown by Lemma B.29 above, the backward induction implications result in an inconsistency at the very start of the game. The contradiction illustrated by Figure 1 for $V < \underline{V}$ highlights the challenges in applying the divinity refinement to our setting due to the endogenous rents obtained by the low type agent from the necessary period of initial pooling. At the same time, Figure 1 does show that for $V > \underline{V}$ the quiet period is sufficiently short that the higher skilled agents become less likely to deviate, as proxied by $d_t(p^H; V) > d_t(p^L; V)$ impying that the equilibrium constructed in Lemma B.29 survives the D1 refinement 4.

Parameters and functional forms. In what follows, we provide additional details on the model parametrization used to obtain Figure 1 – we consider a rather intuitive semi-analytic parametrization of the model in order to maximize tractability and, ultimately, numerical precision. We consider a special case of the discount rate r equal to the public news informativeness λ, i.e., $r = \lambda$, to obtain analytical expressions for the agent's outside option $U(p, k)$. Also, for simplicity, we assume that $L = 0$.[10] None of the previous results relied on the relative comparison of r and λ, so constructing such an example is without loss.[11] Under such parametrization, the expected value to $\theta = 1$ agent from reputation k is equal to

$$
\begin{aligned}
u_1(k) &= \int_0^\infty e^{-rt} \cdot \frac{k}{k + (1-k) \cdot e^{-\lambda t}}\, dt = \int_0^\infty e^{-\lambda t} \cdot \frac{k}{k + (1-k) \cdot e^{-\lambda t}}\, dt \\
&= -\frac{1}{\lambda} \int_0^\infty \frac{k}{k + (1-k) \cdot e^{-\lambda t}}\, de^{-\lambda t} = \frac{1}{\lambda} \int_0^1 \frac{k}{k + (1-k) \cdot x}\, dx \overset{\lambda = r}{=} -\frac{1}{r} \frac{k}{1-k} \ln(k).
\end{aligned}
$$

The value to $\theta = 0$ agent from reputation k is equal to

$$
\begin{aligned}
u_0(k) &= \int_0^\infty e^{-(r+\lambda)t} \cdot \frac{k}{k + (1-k) \cdot e^{-\lambda t}}\, dt = \int_0^\infty e^{-2\lambda t} \cdot \frac{k}{k + (1-k) \cdot e^{-\lambda t}}\, dt \\
&= -\frac{1}{\lambda} \int_0^\infty \frac{k \cdot e^{-\lambda t}}{k + (1-k) \cdot e^{-\lambda t}}\, de^{-\lambda t} = \frac{1}{\lambda} \int_0^1 \frac{k \cdot x}{k + (1-k) \cdot x}\, dx = \frac{1}{\lambda} \int_0^1 \frac{k}{1-k} \frac{k + (1-k) \cdot x - k}{k + (1-k) \cdot x}\, dx \\
&= \frac{1}{\lambda} \cdot \frac{k}{1-k} - \frac{k}{1-k} \cdot \frac{1}{\lambda} \int_0^1 \frac{k}{1-k} \frac{k}{k + (1-k) \cdot x}\, dx \\
&= \frac{1}{\lambda} \cdot \frac{k}{1-k} + \frac{k}{1-k} \cdot \frac{1}{\lambda} \frac{k}{1-k} \ln(k) \overset{\lambda = r}{=} \frac{1}{r} \cdot \frac{k}{1-k} \cdot \left(1 + \frac{k}{1-k} \cdot \ln(k)\right).
\end{aligned}
$$

[10] All our previous results continue to hold in this case as the low type agent is exactly indifferent between staying in the industry and leaving conditional on receiving a bad performance shock.

[11] We have constructed other, purely numerical, examples in which $r \neq \lambda$ and the results of this section are unchanged.

Due to the risk-neutrality of the agent and the linearity of the production function, it follows that as long as the agent's belief about his own ability coincides with clients' belief about his ability, then his expected value is equal to $\frac{k}{r}$:

$$k \cdot u_1(k) + (1-k) \cdot u_0(k) = \frac{k}{r}.$$

Low type's separation interval $[t_1^*, t_2^*]$. The next step is to solve for t_1^* and t_2^*. For an arbitrary q and p solve for t such that

$$\begin{aligned}\pi(q,t) - \pi(p,t) &= rV \\ \frac{q}{q+(1-q)e^{-\lambda t}} - \frac{p}{p+(1-p)e^{-\lambda t}} &= rV \\ q \cdot \left(p + (1-p)e^{-\lambda t}\right) - p \cdot \left(q + (1-q)e^{-\lambda t}\right) &= rV \cdot \left(p + (1-p)e^{-\lambda t}\right) \cdot \left(q + (1-q)e^{-\lambda t}\right).\end{aligned}$$

Simplify terms to obtain

$$\begin{aligned}(q-p) \cdot e^{-\lambda t} &= rV \cdot \left(p + (1-p)e^{-\lambda t}\right) \cdot \left(q + (1-q)e^{-\lambda t}\right) \\ (q-p) \cdot e^{-\lambda t} &= rV \cdot (pq + (q+p-2qp) \cdot e^{-\lambda t} + (1-p)(1-q)e^{-2\lambda t}).\end{aligned}$$

This equation simplifies to a quadratic equation in $e^{-\lambda t}$, given by

$$e^{-2\lambda t} \cdot (1-p)(1-q) + e^{-\lambda t} \cdot \left(q + p - 2qp - \frac{q-p}{rV}\right) + pq = 0,$$

which admits an analytic solution which we denote by $t(p,q)$. Using such notation, we can express in closed form $t_1^* = t(p^L, q_0)$ and $t_2^* = t(p^L, 1)$.

High type's separation time $\bar{t}$. Having characterized t_2^* we can use it to characterize time $\bar{t}$ when the high type separates. Following Lemma B.27 the low type agent must be indifferent between separating at t_2^* and separating at $\bar{t}$:

$$\begin{aligned}U\left(p^L_{t_2^*}, p^L_{t_2^*}\right) + V = &- p^L_{t_2^*} \cdot \int_{t_2^*}^{\bar{t}} e^{-r(t-t_2^*)} \cdot rV\, dt - (1-p^L_{t_2^*}) \cdot \int_{t_2^*}^{\bar{t}} e^{-(r+\lambda)(t-t_2^*)} \cdot rV\, dt \\ &+ p^L_{t_2^*} \cdot \frac{1}{r} + (1-p^L_{t_2^*}) \cdot \frac{1}{r+\lambda} + V,\end{aligned}$$

where $p^L_t \overset{def}{=} \pi(p^L, t)$. By definition of t_2^* in (B.86) and the linearity of $A(\cdot)$ it follows that $p^L_{t_2^*} = 1 - rV$. Then

$$\frac{1-rV}{r} = -(1-rV) \cdot \int_{t_2^*}^{\bar{t}} e^{-r(t-t_2^*)} \cdot rV\, dt - rV \cdot \int_{t_2^*}^{\bar{t}} e^{-(r+\lambda)(t-t_2^*)} \cdot rV\, dt + \frac{1-rV}{r} + \frac{rV}{r+\lambda}$$

$$\begin{aligned}
\frac{rV}{r+\lambda} &= (1-rV)\cdot\int_{t_2^*}^{\bar{t}} e^{-r(t-t_2^*)}\cdot rV\,dt + rV\cdot\int_{t_2^*}^{\bar{t}} e^{-(r+\lambda)(t-t_2^*)}\cdot rV\,dt \\
\frac{r}{r+\lambda} &= (1-rV)\cdot\left(1-e^{-r(t_2^*-\bar{t})}\right) + \frac{rV}{r+\lambda}\cdot\left(1-e^{-(r+\lambda)(\bar{t}-t_2^*)}\right). \\
0 &= \frac{rV}{r+\lambda}\cdot e^{-(r+\lambda)(\bar{t}-t_2^*)} + (1-rV)\cdot e^{-r(\bar{t}-t_2^*)} + \frac{r}{r+\lambda} - (1-rV) - \frac{rV}{r+\lambda} \\
0 &= \frac{rV}{r+\lambda}\cdot e^{-(r+\lambda)(\bar{t}-t_2^*)} + (1-rV)\cdot e^{-r(\bar{t}-t_2^*)} - \frac{\lambda}{r+\lambda} + rV - \frac{rV}{r+\lambda} \\
0 &= \frac{rV}{r+\lambda}\cdot e^{-(r+\lambda)(\bar{t}-t_2^*)} + (1-rV)\cdot e^{-r(\bar{t}-t_2^*)} + \frac{rV(r+\lambda-1)-\lambda}{r+\lambda}.
\end{aligned}$$

The above is a decreasing function in $\bar{t}$, implying that there is a unique solution. Suppose that $r=\lambda$. Then the above is a quadratic equation for $e^{-r(\bar{t}-t_2^*)}$, which simplifies to

$$\frac{V}{2}\cdot e^{-2r(\bar{t}-t_2^*)} + (1-rV)\cdot e^{-r(\bar{t}-t_2^*)} + \frac{V(2r-1)-1}{2} = 0.$$

and admits an analytic solution.

Expected equilibrium payoffs. The expected payoff at $t=0$ to the intermediary-agent pair employing the low type agent is given by $W_0(p^L)$ given by

$$\begin{aligned}
W_0(p^L) = &\int_0^{t_1^*} e^{-rt}\cdot\left(p^L+(1-p^L)\cdot e^{-\lambda t}\right)\cdot\left(A(\pi(q_0,t))-rV\right)dt \\
&+ e^{-rt^*}\cdot p^L\cdot u_1(\pi(p^L,t_1^*)) + e^{-(r+\lambda)t^*}\cdot(1-p^L)\cdot u_0(\pi(p^L,t_1^*)) + V.
\end{aligned}$$

The payoff to the high type in this candidate equilibrium is

$$\begin{aligned}
W_0(p^H) &= \int_0^{\bar{t}} e^{-rt}\cdot\left(A(q_t)-rV\right)dt + e^{-r\bar{t}}\cdot u_1(\pi(p^H,\bar{t})) + V \\
&= \int_0^{t_1^*} e^{-rt}\cdot\left(A(\pi(q_0,t))-rV\right)dt + \int_{t_1^*}^{t_2^*} e^{-rt}\cdot A(\pi(p^L,t))\,dt \\
&\quad + \int_{t_2^*}^{\bar{t}} e^{-rt}\cdot\left(A(\pi(p^H,t))-rV\right)\,dt + e^{-r\bar{t}}\cdot u_1(1) + V.
\end{aligned}$$

Having derives the above expressions analytically, we compute the indifference beliefs $d_0(p^L)$ and $d_0(p^H)$ numerically to obtain Figure 1.

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