The Design of Macroprudential Stress Tests

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Abstract

We study the design of stress tests that provide information about aggregate and idiosyncratic risk in banks' portfolios and impose contingent capital requirements. In the optimal static test, an adverse scenario fails all weak and some strong banks, limiting the stigma of failure. Sequential tests outperform static tests. Under natural conditions, an optimal sequential test consists of a recapitalization followed by a scenario that fails only weak banks, similar to TARP in 2008 followed by SCAP in 2009. Our results highlight the benefits of combining precautionary recapitalization with stress test design, which possibly contributed to SCAP's efficacy and transparency.

Keywords: stress tests, capital requirements, systemic risk, macro-prudential regulation, mechanism design, dynamic mechanisms.

1 Introduction

Over the last decade, stress tests of large financial institutions have become an essential forward-looking tool for bank regulators. Stress testing is used to understand, evaluate, and address the risks posed to the financial system.

There are two key aspects of stress tests. The regulator can manage risk at the individual bank level (the micro-prudential aspect) by making capital requirements conditional on the individual bank's stress test result. Moreover, the stress test aggregates dispersed information across individual bank portfolios and, by doing so, uncovers systemic risks and their implications on the stability of the financial system as a whole (the macro-prudential aspect). In other words, micro-prudential aspects of stress tests are about discovering and responding to idiosyncratic risks (that can spill over to the rest of the system). The macro-prudential aspects are about aggregate risks (for example, discovering a large correlation in individual banks' risk exposures).

The macro-prudential stress test design faces the challenge of informing market participants of underlying systemic risks without causing instability. To achieve this goal, the optimal (static) stress test must be partially informative if conducted on its own to avoid exposing the entire system to a capital shortfall. In this paper, we argue that such a need for opacity evaporates once we allow the regulator to require banks to raise capital before conducting a stress test. We show that such precautionary recapitalization dominates the optimal static test and, moreover, is the optimal sequential intervention in a broad class of dynamic regulatory policies whenever asset riskiness is high relative to trading frictions.

This paper contains three main results. First, we show that the optimal (static) stress test relies on a partially informative adverse scenario that is failed by weak banks and some strong ones too. Such partial informativeness reduces the stigma of failing the stress test and facilitates recapitalization. However, failing strong banks puts excessive limits on their risk-taking and leads to asset misallocation – the test's partial transparency is accordingly costly.

Second, the regulator can generate a net welfare improvement over the optimal stress test by recapitalizing the banks before the stress test, with the benefits being highest during times of high risk. By first implementing stringent capital requirements, the regulator stabilizes banks and is then able to carry out a more informative stress test: stronger balance sheets limit the consequences of failing the stress test and eliminate the need to fail strong banks to support weak ones. Strong banks that pass the stress test take advantage of the slack in the capital constraints by increasing the amount of socially desirable risky investment. The welfare gain from passing all the strong banks in a stress test dominates the welfare cost of tightening capital requirements prior to the test. Surprisingly, by recapitalizing the banks prior to the stress test, the regulator increases the expected capital available to the banks after the test is conducted. The beneficial substitution of precautionary capital requirements followed by a fully informative stress test for an imperfectly informative stress test makes for a powerful argument in favor of stress test transparency while maintaining bank stability. Precautionary recapitalization also improves risk reallocation across banks in good times while reducing fire sales in bad times, which in turn supports higher asset prices ex-ante.

Finally, we consider a broad class of dynamic mechanisms referred to as sequential stress tests. This set includes precautionary recapitalization as a special case. We show that a precautionary recapitalization followed by a single test is optimal when trading frictions are relatively low, or asset riskiness is relatively high. Otherwise, the regulator can further improve welfare by disclosing information and implementing capital requirements over multiple steps.

We model the financial system as a collection of banks and a competitive market of investors. Banks own safe and risky assets and are partially financed by debt obligations. The riskiness of bank assets affects their market price and constitutes the focus of the stress test. The regulator designs the stress test to maximize the expected welfare of all the agents in the financial system net of distress costs that a bank's failure imposes on the broader economy. The (static) stress test consists of first acquiring and disclosing information about the risk of the banks' assets and then setting capital requirements contingent on the shared outcome. We show that the regulator can influence the informativeness of this exercise by publicly choosing a particular scenario, mild or adverse, and evaluating the performance of banks' balance sheets under it. The regulator then uses her supervisory authority to specify enforceable capital requirements, which we identify with risk-weighted capital adequacy ratios (CAR), contingent on the outcome of the stress test. To parsimoniously capture both capital regulation and interbank trade, in our main analysis, we assume that the bank improves its risk-weighted capital ratio only by selling risky assets to the market, but our results hold if the bank can also raise capital via common equity.[1]

The static stress test trades off informed asset allocation with the possibility of socially costly distress stemming from imperfect risk-sharing. A more informative stress test, on the one hand, gives the regulator the flexibility to fine-tune capital requirements freeing up the strong banks to make profitable investments while simultaneously imposing a strict capital requirement for weak banks. On the other hand, negative information released by the stress test increases perceived asset

[1] We place the analysis of equity issuance in Section B.2 of the Online Appendix.

risk and, as a result, depresses market prices, making it harder for weak banks to recapitalize. As we show in Section 3, the optimal static stress test trades off these considerations by relying on an adverse scenario, passing only a fraction of the strong banks, and setting high capital requirements for all those who fail the test. Notably, the stress test's adversity is higher when the financial system is riskier as a whole, setting a higher bar for strong banks to pass. The supervisory authority to set capital requirements is valuable whenever the banks' shareholders, or managers, do not internalize social distress costs to a sufficient extent.

A round of precautionary recapitalization prior to the stress test is beneficial when the financial system's perceived riskiness is high. Precautionary recapitalization enhances risk-sharing and allows for a more informative stress test to follow. That, in turn, improves asset allocation without increasing financial distress costs. The partial irreversibility of the bank's portfolio dynamics due to the disclosure of stress test results is key to the optimality of such multi-step interventions. If the bank first raises capital via asset sales but then passes the stress test, it can reacquire its risky asset. Such round-trip transaction has a lasting effect on the bank's balance sheet as the bank suffers a capital loss: it reacquires the asset at a higher price if it passes the test and is, thus, disclosed as having good quality assets. The strong bank's capital loss is offset by the weak bank's capital gain; the latter ends up selling its assets at a premium during precautionary recapitalization, generating effective risk-sharing between weak and strong banks. However, the strict benefit comes from the regulator's ability to conduct a more informative stress test that fails fewer strong banks and specifies asset holdings that are efficient ex-post. The degree to which the bank's trading actions are irreversible is endogenous to the informativeness of the stress test (a less informative stress test creates smaller variation in asset prices, making it easier to reacquire the asset) and the degree of precautionary recapitalization, both of which are optimally chosen by the regulator. As we show in Section 4, not only does a round of precautionary recapitalization before the stress test dominate the expected welfare under a static stress test, but, under some conditions, it also achieves the global optimum in a class of dynamic mechanisms, which we term "sequential stress tests", in which the regulator can communicate information and set capital requirements over multiple rounds.[2]

Precautionary recapitalization serves two additional benefits. First, as shown in Section 5, it increases the interbank market liquidity because the subsequent stress test fails fewer strong banks. The strong banks that pass the test can buy more risky assets from the banks that fail the test and reduce the need for sales to outside investors, which improves allocation. Second, as we show in Section 6, the increase in the banks' overall liquidity reduces the risks and severity of fire sales, which supports asset prices ex-ante and reduces the amount of capital banks need to raise ex-ante. The optimal sequential stress test can do even better by implementing perfect risk-sharing of idiosyncratic risk across banks and resorting to outside investors only when aggregate risk is sufficiently high.

The beneficial impact of precautionary recapitalization before the stress test underscores the positive effect that equity infusions via the Troubled Asset Relief Program (TARP) had on the efficacy and transparency of the subsequent Supervisory Capital Assessment Program (SCAP) implemented by the U.S. bank regulators during the height of the financial crisis. This mechanism complements the argument made by Faria-e Castro, Martinez, and Philippon (2015) that the optimal stress test informativeness increases with the government's bailout capacity. The bank's ex-ante recapitalization serves as an effective substitute to bailouts while also achieving better risk-sharing and reducing the necessity for providing public funding at an ex-post loss. The bank recapitalization conducted via TARP, which significantly reduced banks' riskiness in its own right, could have been an important factor for why the stress tests administered by the U.S. bank regulators were significantly more informative and compelling than those carried out by their European counterparts during the 2007-2009 financial crisis.

1.1 Related Literature

The canonical trade-off between the allocative efficiency of transparency and the risk-sharing benefits of opacity plays a central role in our analysis and is known as the Hirshleifer (1971) effect.

[2] By revelation principle, an optimal sequential stress test dominates a situation in which the regulator privately communicates information to the bank.

The implications of the Hirshleifer (1971) effect on the design of optimal stress tests were first studied by Goldstein and Leitner (2018).[3] We contribute to the literature by analyzing a joint design of stress tests and capital requirements. In contrast to Goldstein and Leitner (2018) we show that the optimal static stress test fails some strong banks (as opposed to passing some weak banks) when the average quality of banks is sufficiently high due to the complementarity between capital requirements and disclosure of positive information. Moreover, we are the first to consider sequential tests and point out the benefits of precautionary recapitalization prior to a stress test. Surprisingly, this finding circles back to the argument of Hirshleifer (1971) that trade followed by information disclosure and subsequent reallocation may be beneficial in the context of consumption risk-sharing. The regulator's role in requiring recapitalization is important as bank shareholders may not follow socially optimal actions if there is a wedge between social and private benefits of recapitalization, as highlighted by Williams (2017) in the context of static tests.

Ong and Pazarbasioglu (2014) point out the efficacy and transparency of the SCAP in the U.S., relative to its counterpart, the Committee of European Banking Supervisors (CEBS) in the E.U. in 2009 and 2010. Faria-e Castro, Martinez, and Philippon (2015) show that the optimal stress test may have been more informative in the U.S. due to the greater bailout capacity available to its regulators at the time. In our model, a lower bailout cost, proxied by a lower social distress cost, also results in an incrementally more informative optimal static test. Such a stress test could, however, require ex-post government bailouts. In contrast, precautionary recapitalization leads to a more informative test but reduces the need for government funds, acting as a form of market insurance, advocated for by Kashyap, Rajan, Stein, et al. (2008) to manage systemic risk. As predicted by the optimal dynamic mechanism in our model, both the efficacy and transparency of SCAP can, at least in part, be attributed to the U.S. regulators' successful efforts in recapitalizing the banks ex-ante via the TARP program which, as documented by Veronesi and Zingales (2010), was responsible for significantly reducing bank CDS spreads.[4] Such implementation of the optimal

[3] Monnet and Quintin (2017) study a similar trade-off in the context of securitization.

[4] In the model, banks improve their capital ratios via asset sales, rather than equity issuances. This is motivated by the leverage ratchet effect of Admati, DeMarzo, Hellwig, and Pfleiderer (2018) and the added benefit of incorporating interbank trade in the model. Veronesi and Zingales (2010) point to the qualitative similarities between asset sales and equity infusions in the context of the Troubled Asset Relief Program, but acknowledge that equity infusions achieve greater efficiency than asset sales. We show in Appendix B.2 that our static and dynamic results are robust

mechanism via two distinct, but sequential, regulatory interventions highlights the importance of the joint effects of contemporaneous bank regulations, as emphasized by Greenwood, Stein, Hanson, and Sunderam (2017).

We start by assuming that the regulator can commit to an arbitrary information disclosure policy, as in Kamenica and Gentzkow (2011), but provide an implementation of the optimal policy via subjecting the banks to public adverse stress scenarios, similar to the idea in Parlatore and Philippon (2018) that greater scenario adversity reduces the informativeness of the outcome.[5] In light of the portfolio choice problem solved by the regulator when setting capital requirements, our model provides a novel setting for considering the costs and benefits of disclosing information to an active market, contributing to Goldstein and Yang (2017) who focus on the CARA-normal model. The optimal information disclosure in our model is asymmetric and skewed towards disclosing good, i.e., low-risk states, as their relatively higher Sharpe ratio allows the regulator to relax capital requirements by a greater margin ~~in that state~~ and achieve higher welfare.

Our results on the optimality of sequential testing stem from partial irreversibility of the impact of trade on banks' portfolios in the presence of sequential information disclosures. While disclosure of a positive piece of information after a negative one could have a completely offsetting effect on the price, interim trade encodes the *path* of prices into the bank's portfolio. In this respect, our paper is related to Grenadier, Malenko, and Malenko (2016), Orlov, Skrzypacz, and Zryumov (2020), and Malenko and Tsoy (2019) who exploit irreversibility of time in the context of stopping games.

A large literature explores the effects of regulatory disclosures in a crisis. Bouvard, Chaigneau, and Motta (2015) show that, in the face of a looming bank run, the regulator can reduce inefficient liquidations by disclosing them prior to the run materializing. Shapiro and Skeie (2015) show the regulator could provide markets with information about bank balance sheets via costly bailout actions. Gick and Pausch (2013) show the benefits of credible but partial information disclosure, while Inostroza and Pavan (2018) emphasize the importance of decentralized communication in

to equity issuance. See Philippon and Schnabl (2013) for an in-depth analysis of optimal recapitalizations outside the context of stress tests.

[5]Flannery, Hirtle, and Kovner (2017), Petrella and Resti (2013), Ong and Pazarbasioglu (2014), and Peristiani, Morgan, and Savino (2010) provide empirical evidence that stress test disclosures in the U.S. via SCAP in 2009 and in the E.U. via CEBS in 2010 contained new information manifested through abnormal returns and trading volume.

environments with dispersed creditors. Importantly, these papers focus on the optimal way to manage a crisis, whereas our model offers guidance on how the regulator can avoid it in a precautionary way via informative stress tests and capital requirements. Inostroza (2020) shows that, if the bank's short- and long-term creditors are segmented, the bank can raise more funds from long-term creditors if the regulator subsequently conducts a stress test that minimizes the run of short-term creditors in light of liquidity risk realization. Our results are qualitatively different as they speak to the optimality of sequential information disclosure and balance sheet adjustments, even though all parties are long-lived, and there is only one source of uncertainty, which is optimally resolved through the regulator's sequential disclosures.

Our analysis of multiple banks in Section 5 shows the distinct manifestations of aggregate and idiosyncratic risks in the design of the optimal stress test. Alvarez and Barlevy (2015) show that in the event of severe contagion, the banks' private incentives to disclose the quality of their balance sheet may be insufficient, and the regulator can contribute by revealing systemically important institutions. Huang (2020) considers a network model of banks and shows that the optimal stress test induces a correlation structure across stress test outcomes that benefits systemically important banks, allowing them to raise the necessary capital while undermining recapitalization of peripheral banks. We show that precautionary recapitalization can be even more valuable in the presence of interbank trade since it increases the subsequent stress test's informativeness and, consequently, improves the liquidity of the interbank market.

2 Baseline Model

The model has three time periods indexed by $t \in \{0, 1, 2\}$. There is a numeraire good, referred to as cash, and two types of tradable assets. The first asset is safe, modeled as a riskless bond with a unit face value maturing at $t = 2$. To economize notation, we, without loss, normalize all parties' discount rate to 0. The second asset is risky and pays an uncertain amount $X \sim U[\theta, 1]$ at $t = 2$. The random variable $\theta \in \{0, 1\}$ captures the quality of the risky asset. The risky asset is payoff equivalent to the safe asset if $\theta = 1$, but not if $\theta = 0$.[6] Risky asset quality θ is realized at $t = 1$, ahead of the actual cash flow realization X at $t = 2$, but not directly observed by market participants. There is common prior $\mathrm{P}_0(\theta = 1) = \pi_0$. The economy is comprised of the financial sector, which, for now, we identify with a single bank (or a continuum of identical banks) and a competitive market of risk-neutral investors. The bank is the natural holder of the risky asset and, whereas the risky asset pays X if the bank holds it at $t = 2$, it only pays δX if outside investors hold it at $t = 2$, where $\delta < 1$.[7]

The bank's starting portfolio at $t = 1$ is given by $b > 0$ units of the safe asset, equivalent to cash, and $a > 0$ units of the risky asset. The bank can trade the risky asset in the market in period $t = 1$, before the risky cash flow X is realized. The market price of the risky asset at $t = 1$ is equal to $\delta \mathrm{E}[X]$, where the expectation is taken with respect to all information available to the market about X, i.e., θ, at the time of the trade in $t = 1$. The bank also has a liability in the amount d maturing in period $t = 2$. We assume it is deterministic and refer to it as debt repayment, but the analysis can be extended for a stochastic realization of d stemming from the settlement of other financial contracts, such as options or credit default swaps. If the bank repays its period $t = 2$ liability d, the remaining cash flows are distributed to the bank's shareholders. If the bank is unable to pay d in the second period, then it enters distress and imposes social cost. To account for the multiple channels through which the bank's distress is socially costly, we model it in reduced form via an increasing convex function $c(\cdot)$ of the bank's cash shortfall at $t = 2$, i.e., of how much additional cash the bank would need to repay its liability d when it comes due.[8] If the bank enters period $t = 2$ with portfolio $(\hat{b}, \hat{a})$, the realized social welfare is

$$\textit{Social Welfare} = \underbrace{b + \hat{a} \cdot X + (a - \hat{a}) \cdot \delta X}_{(i)} - \underbrace{c\big(\max\big[d - \hat{b} - \hat{a} \cdot X,\ 0\big]\big)}_{(ii)}, \tag{1}$$

where (i) is the joint welfare of the bank's shareholders and creditors, as well as capital market

[6]We assume there is no cash flow risk in state $\theta = 1$ to make the contrast between $\theta = 0$ and $\theta = 1$ stark. All of our results hold if, conditional on $\theta = 1$, the payoff X is risky, e.g., $X \sim U[\bar{\theta}, 1]$ with $\bar{\theta} \in (0, 1)$.

[7]See DeMarzo and Duffie (1999) and Duffie, Gârleanu, and Pedersen (2005) for a similar discount specification.

[8]Bank failure may lead to financial contagion Allen and Gale (2000)), fire sales and liquidity spirals (Caballero and Simsek (2013), Brunnermeier and Pedersen (2009)), and ultimately a contraction of credit for the real economy (Ivashina and Scharfstein (2010)). Alternatively, the regulator may be required to bailout the bank to avoid its failure (Faria-e Castro et al. (2015)). We assume that $c(\cdot)$ is convex to capture the increasing marginal cost of either real effects of distress or bailout funds in the severity of the bank's bankruptcy.

investors and (ii) is the realized distress cost. The liability d nets out in (i) since it is a transfer from the bank's shareholders to creditors, but not in (ii) where it affects the potential severity of the bank's cash shortfall. The bank's rebalancing from its starting portfolio (b, a) to portfolio $(\hat{b}, \hat{a})$ in $t = 1$, affects the allocative efficiency of the risky asset in (i) and distress costs in (ii).

Capital Adequacy Ratio. The regulator's objective is to maximize the expected value of social welfare in (1), as the bank's shareholders do not internalize the social costs of distress. Traditionally, this goal has been achieved using various forms of capital and liquidity requirements, aimed at managing the bank's solvency risk, by limiting the banks' risk-taking given its liabilities.[9] Our model features no maturity mismatch, and, consequently, both the capital and liquidity regulation can be identified with the bank's Capital Adequacy Ratio (CAR), defined in (2) as the ratio of the marked-to-market value of the bank's assets net of liabilities to the risk-weighted value of assets. The bank's initial CAR can be written as

$$R_0 \overset{def}{=} \frac{b + a \cdot \delta \mathrm{E}[X] - d}{0 \cdot b \cdot 1 + 1 \cdot a \cdot \delta \mathrm{E}[X]}. \qquad \textit{(Capital Adequacy Ratio)} \tag{2}$$

Whenever the market value of the bank's assets is above its liabilities, i.e., $b + a \cdot \delta \mathrm{E}[X] \geq d$ a reduction in the risky asset holdings improves the bank's capital ratio: retention of $A < a$ units of the risky asset by selling $a - A$ of them to the market increases its safe asset holdings to $B = b + (a - A) \cdot \delta \mathrm{E}[X]$ and does not affect the numerator in (2), but reduces the value of risk-weighted assets in the denominator of (2). Inversely, a bank can to attain a prescribed CAR equal to R, by rebalancing its portfolio to

$$A \overset{def}{=} \frac{b + a \cdot \delta \mathrm{E}[X] - d}{R \cdot \delta \mathrm{E}[X]}, \qquad B \overset{def}{=} d - \frac{1 - R}{R} \cdot \big(b + a \cdot \delta \mathrm{E}[X] - d\big). \tag{3}$$

Static Stress Test. In the midst of the financial crisis of 2007-2009, the uncertainty about the U.S. financial system's health led the Federal Reserve to supplement traditional regulation with

[9] In the U.S., the Federal Reserve enforces minimal liquidity and capital ratios for SIFIs stipulated by Basel III and Dodd-Frank Act. While the minimum common equity tier 1 ratio is fixed at 4.5%, the Fed requires the banks to hold additional Stress Capital Buffer, which is determined from the stress test results. For details see `https://www.federalreserve.gov/newsevents/pressreleases/bcreg20200810a.htm`.

a stress test combined with capital requirements contingent on its result. The test provided the broad market with information about the riskiness of the banks' portfolios by evaluating their performance under a hypothetical macro-economic scenario.[10] The choice of this scenario provided the regulator with a degree of flexibility about what kind and how much information the stress test outcome conveyed, which, in the context of our model, we identify with a signal S about θ.[11] The regulator then used its supervisory authority to tighten the capital requirements for banks which performed poorly in the stress test. Such contingent capital requirements are then naturally captured by a target capital ratio $R(s)$ which depends on the outcome $S = s$.[12]

Definition 1. *A (static) stress test $\mathcal{S} = \{S, R(\cdot)\}$ is a signal S, correlated with θ, and a capital adequacy ratio $R(s)$ contingent on the stress test outcome $S = s$.*

- *A stress test is pass/fail if it induces a binary outcome $S \in \{pass, fail\}$ and the capital adequacy ratio upon failing the test is higher than upon passing it, i.e., $R(fail) > R(pass)$.*

A (static) stress test $\mathcal{S}^* = \{S^*, R^*(\cdot)\}$ is *optimal* if it maximizes the expected social welfare

$$\mathcal{S}^* \in \arg\max_{\mathcal{S}} \mathrm{E}\left[b + A^*(S) \cdot X + (a - A^*(S)) \cdot \delta X - c\left(\max\left[d - B^*(S) - A^*(S) \cdot X,\ 0\right]\right)\right]. \quad (4)$$

The regulator designs and commits to a stress test ex-ante, i.e., at $t = 0$, before the riskiness θ is realized, capturing the fact that she has no private information about θ relative to the market when committing to a test.[13] Figure 1 captures the timing of the model.

It is convenient to define a particular class of pass/fail stress tests, motivated by the intuition behind adverse scenarios, used by both the U.S. and E.U. regulators, which always fail the weak, $\theta = 0$, bank and probabilistically fail the strong, $\theta = 1$. The more adverse the stress test scenario

[10] We show explicitly in Section 6 that the stress test is warranted as the regulator has a comparative advantage of identifying bank risks by virtue of being able to observe the cross-section of banks' portfolios.

[11] We provide an explicit mapping between signal S and stress scenarios in Section 2.1.

[12] State contingent capital requirements depend on S and not θ to ensure they do not convey any information to the market above and beyond the stress test outcome. This is without loss of generality since if we allowed $R(\cdot)$ to depend on θ beyond what is revealed by the signal S, the rational market would make the correct inference and price the risky asset based on both S and $R(\cdot)$, so we would simply redefine the signal to include the information contained in $R(\cdot)$. In other words, we implicitly assume that the regulator cannot make private recommendations to the bank in a way that would allow the bank to sell some assets before the market learns about the recommendation. Such recommendations are suboptimal in the context of sequential stress tests below.

[13] It does not matter whether the bank knows (but cannot credibly disclose) θ due to the regulator's supervisory authority to enforce capital requirements.

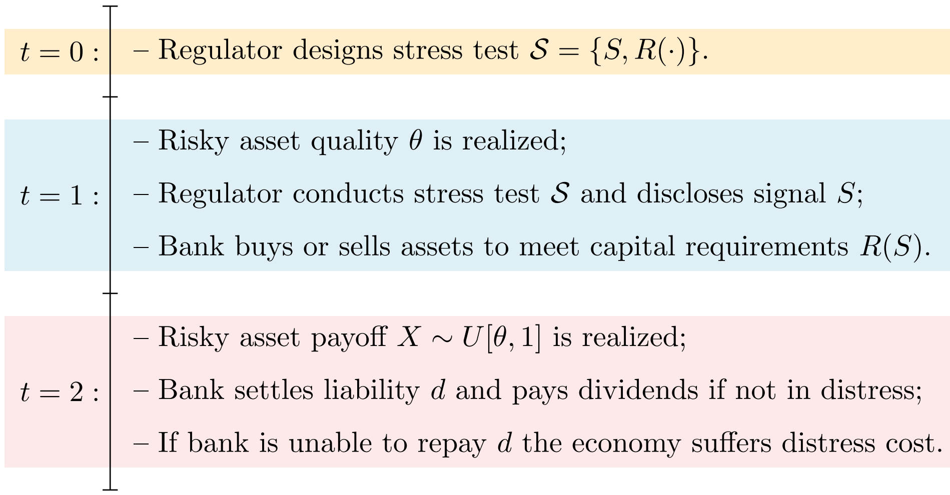


Figure 1: Timing of the stress test and cash flows in the model.

is, the more likely even a bank with good assets is to fail it, hence, the higher is the rate of false negatives generated by the stress test, and the lower is the overall informativeness of the stress test. It turns out that such pass/fail tests feature prominently in the characterization of the optimal stress test.[14]

Definition 2. *A stress test implements an adverse scenario if it is a pass/fail test with zero false positive and some false negative outcomes, i.e.,*

$$\mathrm{P}(S = \mathit{pass} \,|\, \theta = 0) = 0 \quad \mathit{and} \quad \mathrm{P}(S = \mathit{fail} \,|\, \theta = 1) > 0.$$

The regulator's joint design of the informativeness of the stress test signal S and associated capital requirements $R(S)$ leads to an economic trade-off. On the one hand, a highly informative test allows the regulator to impose stringent capital requirements only if they are truly needed, i.e., if $\theta = 0$. On the other hand, revealing that $\theta = 0$ leads to a drop in the market price of the risky asset, reducing the market value of the bank's portfolio in this state and making its distress unavoidable.

Sequential Stress Test. During the 2007-2009 financial crisis, the financial regulators around the world implemented several programs aimed at stabilizing the financial system. One of the most successful such programs in the U.S. was the recapitalization of the largest banks in October

[14] We provide a mapping from such pass/fail tests to hypothetical adverse scenarios in Section 2.1.

2008 under the Troubled Asset Relief Program (TARP).[15] The Supervisory Capital Assessment Program was conducted shortly after the implementation of TARP in February of 2009 and provided a highly informative insight into the banks' riskiness, requiring some banks to raise additional capital, all without causing financial disruptions. In light of the possibility of such *precautionary* recapitalization prior to the stress test, it is natural to consider their effect in conjunction with the stress test design. To do so formally, we extend definition 1 to a (very) broad class of mechanisms that allow the regulator to recapitalize the bank over multiple steps and share additional information about θ with the market participants at each of these steps.

Definition 3. *An N-step sequential stress test $\mathcal{S} = \{S_n, R_n(\cdot)\}_{n=1}^N$ is a sequence of signals $\{S_n\}_{n=1}^N$ and contingent capital ratios $\{R_n(\cdot)\}_{n=1}^N$. At each step $n \in \{1, \ldots, N\}$, the regulator first discloses the stress test outcome $S_n = s_n$ and the bank is required to rebalance its portfolio to meet the new capital requirement $R_n(s_1, \ldots, s_n)$.*

For expositional convenience, we assume the sequential stress test is completed within period $t = 1$, similar to Figure 1, and we abstract from the delay costs associated with the multiple steps.[16] A sequential stress test is optimal if the bank's resulting portfolio, consisting of B_N units of safe asset and A_N units of the risky asset, maximizes the expected social welfare in (4).

Troubled Asset Relief Program (TARP) did not reveal any information about the bank assets and simply increased banks' capital ratios, hence, we can formalize it as $S_1 = \emptyset$ and $R_1(\emptyset) > R_0$. The Supervisory Capital Assessment Program contained a much more informative outcome $S_2 \in \{pass, fail\}$ and associated capital requirements $R(S_2)$. The sequential implementation of these two policies is naturally embedded in the framework of sequential stress tests.

Definition 4. *Precautionary recapitalization followed by a stress test is a two-step sequential stress test $\mathcal{S} = \{S_i, R_i(\cdot)\}_{n=1}^2$ with $S_1 = \emptyset$ and $R_1 > R_0$.*

Surprisingly, such precautionary recapitalization followed by the stress test may achieve the maximum expected welfare among *all* sequential stress tests, as we show in Section 4.1.

[15] A similar program was at play in the U.K. in the summer of 2008. See Veronesi and Zingales (2010) for details.

[16] This is an acceptable approximation as the optimal sequential stress test often has only two steps, and the banks make a significant improvement to their capital ratios at the first step, limiting the carry-over of risk across periods.

2.1 Discussion, Interpretation, and Implementation

So far, we have assumed that the only information market participants learn about θ comes from the regulatory stress test. This assumption is founded on the idea that the financial regulator is able to evaluate systemic risks by collecting proprietary information about bank portfolios. First, the regulator has access to a cross-section of different bank portfolios and identifies the sources of systemic risk in a forward-looking way based on the commonality of their exposures – we show this explicitly in Section 6. Second, the regulator may be better informed about the state of the macroeconomy, leading it to better evaluate the systematic exposure of the banks' portfolios. Finally, even if the bank were better informed about its assets than the regulator, adverse selection might preclude the risky asset's price from incorporating this information. The stress test alleviates this problem both via the credible public disclosure, i.e., certification, of risky asset quality, which reduces the information asymmetry, *and* the publicly observable capital requirements, which alleviate the adverse selection problem in the market by disallowing the bank to strategically retain the high-quality asset.

We assume that banks raise capital by selling assets to the market, rather than diluting existing shareholders, consistent with the leverage ratchet effect of Admati, DeMarzo, Hellwig, and Pfleiderer (2018). We show in Section B.2 of the Online Appendix that our results are unchanged if we focus on recapitalizations via secondary equity issuance. Asset sale changes the denominator of the capital ratio, while equity issuance affects the numerator in (2). We focus on the former in the main text as, in the presence of multiple banks in the economy, it allows us to model an active interbank market and study the implications it has on optimal stress test design in Sections 5 and 6. Despite both capital and liquidity requirements being equivalent in our baseline setting, the introduction of interbank trade allows banks to share liquidity, paving way for liquidity to have a distinct role in the optimality of precautionary recapitalization and sequential tests more generally.

Stress Test Scenarios. The stress test's informativeness can be naturally mapped to the adversity of the stress test scenarios used by the Federal Reserve and the European Central Bank. To do so in the context of our stylized model, we relate the payoff of the risky asset X to the already introduced asset quality θ realized at $t = 1$, but also an exogenous shock Z realized at $t = 2$. The risky asset of quality θ has an unobservable *stochastic* distress threshold z_θ such that, whenever shock Z is less than z_θ, the risky asset pays full value 1 and whenever shock Z exceeds the distress threshold z_θ the asset loses its value and generates a cash flow in $[0, 1]$. Formally,

$$X(Z, \theta) = 1 \cdot \mathbb{1}\{Z \leq z_\theta\} + U[0, 1] \cdot \mathbb{1}\{Z > z_\theta\}. \tag{5}$$

We assume the high quality asset $\theta = 1$ has a distress threshold $z^1 > 1$, while the low quality asset $\theta = 0$ has a distress threshold $z^0 < -1$. The shock Z is drawn at $t = 2$ from a distribution with support on $[-1, 1]$[17]. The regulator does not observe the shock Z prior to its realization, but can evaluate payoff $X(z, \theta)$ in (5) under a hypothetical realization $Z = z$ at $t = 1$, before the actual shock Z is realized at $t = 1$. Very mild ($z \ll -1$) and very severe ($z \gg 1$) stress scenarios reveal little information since the bank either passes or fails such a scenario regardless of the underlying θ, i.e., regardless of whether a bank has high or low-quality assets. However, tests of intermediate severity allow the regulator to distinguish between high and low quality assets. The severity of the stress scenario z determines not only the informativeness of the stress test outcome but also the type of errors that the stress test produces. For example, mild stress scenarios with $z < -1$ pass some of the banks even if $\theta = 0$, thus generating partially informative signals with type 1 errors (false positives). Such scenarios uncover that the asset is of low quality only if $z^0 < z$, but letting low quality assets, characterized by a greater distress threshold $z^0 > z$, look just as well as the high quality assets. On the other end of the spectrum, extremely severe stress scenarios with $z > 1$ generate partially informative tests with type 2 errors (false negatives). They reveal the high quality asset whenever $z^1 > z$, but high quality assets with $z^1 < z$ perform poorly in the test similar to the low quality assets. It is natural to refer to *a scenario as adverse if* $z \geq 1$, leading the weak bank to always fail it. Increasing the severity of the scenario from $z = 1$ to $+\infty$ increases the stress test's failure rate but does so by failing more $\theta = 1$ banks. As a result, failing an adverse scenario

[17]The future distribution of shocks satisfies $\mathrm{P}(z_0 < Z < z_1) = 1$, i.e., the low quality asset always pays $U[0, 1]$ and the high quality asset always pays 1. As a result, stress scenarios with $z < -1$ and with $z > 1$ are outside of the support of future shocks Z. Condition $\mathrm{P}(z^0 < Z) = 1$ is inconsequential and can be relaxed within our model. Condition $\mathrm{P}(Z < z^1) = 1$ is necessary due to assumption that risky asset quality can take, at most, two values $\theta \in \{0, 1\}$ in our model. This condition can be relaxed in the version of this model where the asset quality θ is continuous.

does not impose as much stigma as failing a mild scenario. The intuition for such, admittedly stylized, scenario specification is consistent with Parlatore and Philippon (2018), who show that more adverse scenarios reduce the informativeness of the stress test outcome in a Gaussian setting. In contrast, the mild/adverse scenarios above provide an asymmetric pass-fail structure, skewing the distribution of posteriors up (down) in the case of an adverse (mild) scenario.

Bank Shareholder Preferences. We do not focus on the private incentives of the banks in the portfolio choice problem . As the regulator's objective coincides with social welfare, and the bank's shareholders do not internalize all of the social distress costs, their private cost of distress is always weakly lower than the social one. This assumption alone is sufficient for the majority of the analysis, leading the bank shareholders to always prefer more risky assets to less and, thus, making the capital adequacy requirement bind when the risks are high.[18]

3 Static Stress Tests

The stress test outcome conveys an informative signal S about the riskiness of the bank's asset, which maps to a posterior belief

$$\pi(s) \stackrel{def}{=} \mathrm{P}(\theta = 1 \,|\, S = s)$$

shared by all market participants. For example, under an adverse scenario test, the posterior belief is $\pi(pass) = 1$ and $\pi(fail) < \pi_0$. Posterior belief $\pi(S)$ affects the price $p(S)$ of the risky asset via

$$p(S) \stackrel{def}{=} \delta \cdot \mathrm{E}\left[X|S\right] = \delta \cdot \Big[\underbrace{\mathrm{E}\left[X|\theta=1\right]}_{=1} \cdot \underbrace{\mathrm{P}(\theta=1|S)}_{=\pi(S)} + \underbrace{\mathrm{E}\left[X|\theta=0\right]}_{=1/2} \cdot \underbrace{\mathrm{P}(\theta=0|S)}_{=1-\pi(S)} \Big] = \delta \cdot \frac{1+\pi(S)}{2}.$$

Belief $\pi(S)$ captures the knowledge about state θ and is important in two ways. First, it helps the regulator determine asset allocation to mitigate distress costs – an allocation effect. Second, it affects the value of the bank's initial portfolio through price $p(S)$ – a wealth effect. To illustrate these two distinct channels, it is useful to consider the case when all of the bank's starting wealth is in the safe asset, i.e., the bank has no risky asset. Denote the bank's (constant) starting wealth

[18]Corollary 2 provides a simple illustration of how the conflict between shareholder and the regulator makes enforcing capital requirements valuable.

by w and define by $V(\pi, w)$ to be the regulator's expected payoff given belief π about θ and the bank's starting wealth w:

$$V(\pi, w) \overset{def}{=} \max_{\hat{b},\hat{a}} \ \mathrm{E}_{\pi} \left[w + \hat{a}(1-\delta) \cdot X - c\Big(\max\Big[d - \hat{b} - \hat{a} \cdot X, \ 0 \Big] \Big) \right] \tag{6}$$

subject to a no borrowing constraint $\hat{b} \geq 0$, a no short sales constraint $\hat{a} \geq 0$, and a budget constraint $\hat{b} + \hat{a} \cdot \delta \frac{1+\pi}{2} \leq w$. Viewed from this perspective, the bank's starting wealth is independent of the market price of the risky asset and the regulator's welfare increases with full information disclosure since it allows the regulator to impose strict capital requirements only if $\theta = 0$.

Lemma 1 (Benefit of Information, Allocation Effect)**.** *For any convex cost function $c(\cdot)$ and any starting wealth w, the regulator's value function $V(\pi, w)$ is strictly increasing in π. Moreover, full disclosure of θ dominates non-disclosure for any prior π, as captured by*

$$V(\pi, w) \leq \pi \cdot V(1, w) + (1 - \pi) \cdot V(0, w).$$

The cost of disclosing information about θ stems form the bank's exposure to the price of the risky asset. The prospect of distress costs makes the regulator averse to fluctuations in the bank's wealth.

Lemma 2 (Cost of Information, Wealth Effect)**.** *For any convex cost function $c(\cdot)$ and any belief π, the regulator's value function $V(\pi, w)$ is weakly concave in starting wealth w.*

The bank's starting position in the risky assets leads the market value of its portfolio to fluctuate in response to new information about θ, captured by the identity $w = b + a \cdot \delta \frac{1+\pi}{2}$. A more informative stress test that generates a mean-preserving spread in π, thus increasing the riskiness of the bank's starting portfolio. This exposes the regulator to greater distress costs.

3.1 Optimal Static Stress Test

The regulator designs the stress test to balance the cost of transparency, entering through the wealth effect in Lemma 2, with the benefit of transparency, entering through the allocation effect in Lemma 1. Given belief π about θ, the expected social surplus is given by

$$V(\pi) \overset{def}{=} V\left(\pi, \ b + a \cdot \delta \frac{1+\pi}{2}\right).$$

Define by π_{DF} to be the lowest belief about θ such that the bank can avoid distress with certainty at $t = 2$ were it to sell all of its risky asset to the market at $t = 1$

$$\pi_{DF} \stackrel{def}{=} \min \left\{ \pi \geq 0 : \quad b + a \cdot \delta \frac{1 + \pi}{2} = d \right\}.$$

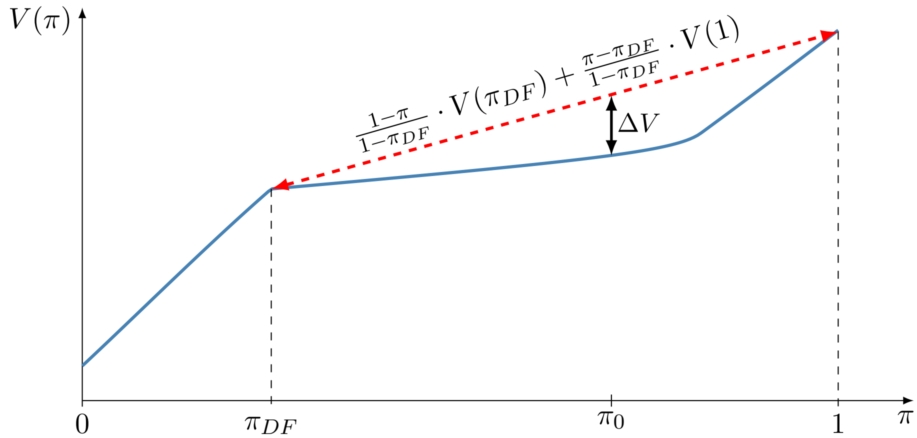


Figure 2: Expected value $V(\pi)$ (solid blue) line as a function of belief π. Concavification of $V(\pi)$ (dashed red) over $[\pi_{DF}, 1]$ line is the expected social value of an adverse pass-fail signal S with posteriors in $\{\pi_{DF}, 1\}$. The difference ΔV captures the incremental social value of such a signal. Parameters: $b = 0.25$, $a = 1$, $d = 0.75$, $\pi_0 = 0.7$, $\delta = 0.8$, $c(x) = 4 \cdot \max(x, 0)$.

Figure 2 illustrates that $V(\pi)$ is convex for $\pi \geq \pi_{DF}$, i.e., when the $t = 1$ market value of the bank's assets exceeds its liabilities. In this region the allocation effect, captured by Lemma 1 dominates the wealth effect as, by setting efficient capital requirements the regulator can avoid bank distress and the associated costs. This implies that for $\pi_0 \geq \pi_{DF}$, the regulator can improve upon the no-information value $V(\pi)$ by disclosing some information about θ. Figure 2 illustrates the expected benefit the regulator obtains from an adverse pass-fail signal S with the failing posterior of π_{DF}: the marginal gain from relaxing the capital constraint and allowing the bank to hold the efficient amount of the risky asset, conditional on passing the stress test, exceeds the marginal welfare loss from a larger asset sale during the bank's recapitalization, conditional on failing the stress test. Proposition 1 shows the optimality of adverse pass-fail stress test structure that discloses *at least* the information between π_{DF} and 1, regardless of the cost function $c(\cdot)$ as long as $\pi_0 \geq \pi_{DF}$.[19]

[19]Proposition 1 characterizes the optimal stress test when $\pi_0 \geq \pi_{DF}$. If $\pi_0 < \pi_{DF}$ distress in unavoidable and the optimal stress test critically depends on the shape of the cost function $c(\cdot)$. For example, if $\pi_0 < \pi_{DF}$, then the optimal stress test reveals no information if $c(\cdot)$ is sufficiently convex.

Optimal capital requirements are pinned down by the optimal risky asset retention $A(\pi)$ that maximizes the expected welfare in (6), given the bank's wealth is given by $w = b + a \cdot \delta \frac{1+\pi}{2}$.

Proposition 1 (Optimal Static Stress Test)**.** *Suppose $\pi_0 \geq \pi_{DF}$. The optimal static stress test is an adverse pass/fail test characterized by failing belief $\pi^* \leq \pi_{DF}$ such that*

$$\mathrm{P}(\theta = 1 \,|\, S = \textit{fail}) = \pi^*.$$

The optimal capital adequacy ratio imposed on the bank if it fails the stress test is

$$R^* = R\left(\pi^*\right) \stackrel{def}{=} \frac{b + a \cdot \delta(1+\pi^*)/2 - d}{A\left(\pi^*\right) \cdot \delta(1+\pi^*)/2}.$$

Proposition (1) shows that, if $\pi_0 \geq \pi_{DF}$, then the optimal stress test can be implemented via a pass/fail message. With probability $\frac{\pi_0 - \pi^*}{1-\pi^*}$ the stress test passes the bank, revealing that the risky asset is of high quality, and allows the bank to not only hold its initial position but even to expand its risky holdings – a rebalancing shareholders are happy to do given the positive expected return. With probability $\frac{1-\pi_0}{1-\pi^*}$, the optimal stress test fails the bank and requires it to recapitalize. The optimal stress test features false-negative errors: it always fails the bank if $\theta = 0$ and, in addition, it fails the bank if $\theta = 1$ with conditional probability $\frac{1-\pi_0}{1-\pi^*} \cdot \frac{\pi^*}{\pi_0}$. False-negative errors introduce cross-state subsidization by inefficiently restricting the bank's portfolio if $\theta = 1$ while simultaneously ensuring that, even if $\theta = 0$, the price of the risky asset is sufficiently high that recapitalization can, at least partially, mitigate the distress costs. An immediate corollary of Proposition 1 is:

Corollary 1 (No Solvency Constraints)**.** *Suppose the capital adequacy ratio of the bank's initial portfolio is weakly positive even when the risky asset price is $p = \delta \cdot \mathrm{E}\left[X|\theta = 0\right]$, i.e., $\pi_{DF} = 0$. Then, the optimal stress test fully reveals θ.*

If $\pi_{DF} \leq 0$, then the value of the bank's portfolio is high enough to avoid distress even if state $\theta = 0$ were to be revealed. In this case the allocation effect makes cross-state risk-sharing suboptimal and the regulator chooses a fully transparent stress test that features no false negatives. If the state turns out to be $\theta = 0$, then the regulator fails the bank and imposes a capital adequacy ratio $R(0)$.

3.2 Properties of the Static Stress Test

The interaction of optimal information disclosure, i.e., scenario choice, and design of capital requirements can lead to an overall reduction in bank distress risk relative to solely managing information or setting capital requirements.

Lemma 3 (Risk-free Stress Test). *Suppose $\pi_0 \geq \pi_{DF} \in (0,1)$. The optimal static stress test makes the bank avoid distress with certainty, i.e., $\mathrm{P}(\theta = 1 | S = fail) = \pi_{DF}$, if and only if the marginal cost of distress is sufficiently high, i.e., $c'(0) \geq \bar{c}$, where $\bar{c}$ is increasing in liability d and discount δ.*[20]

Lemma 3 highlights that safe recapitalization is optimal when the marginal distress cost is sufficiently large *and* the bank is not overly leveraged. Safe recapitalization, however, might be socially *sub-optimal* even if it is feasible, i.e., $\pi_0 > \pi_{DF}$, as it requires too much misallocation. The optimal stress test allows the bank to enter distress with positive probability either when the welfare gains from holding the asset by the bank are higher, as captured by a lower δ, or when avoiding distress requires too much asset sales, as captured by high d. Surprisingly, when $c'(0) \geq \bar{c}$ the optimal stress test is safer than both a fully opaque ($S = \emptyset$) and a fully informative ($S = \theta$) stress tests that may result in bank distress when the asset quality is low ($\theta = 0$), highlighting the complementarity between information/scenario choice and capital requirements.[21]

Supervisory authority allows the regulator to impose minimum capital ratios that the bank must abide by. As to be expected, this is valuable whenever the bank shareholders or managers do not internalize the social distress costs to a sufficient extent.

Corollary 2 (Regulatory Role of Capital Requirements). *Suppose $c'(0) \geq \bar{c}$ holds and the optimal stress test is risk-free. Setting capital requirements is not necessary to implement the optimal stress test if and only if bankruptcy cost function of the bank's shareholders $c'_S(0)$ also satisfies $c'_S(0) \geq \bar{c}$.*

If the stress test did not impose capital requirements, the bank shareholders have an incentive to follow the regulator's portfolio recommendations if and only if their private cost of distress is

[20] A closed form upper bound for $\bar{c}$ is $\frac{1-\delta}{3} \cdot \frac{a}{b+a\delta-d} \cdot \frac{a}{d-b} \cdot \left(\frac{d-b}{a} + \frac{b/\delta+a}{b+a\delta-d}\right)$, illustrating its monotonicity in d and δ.

[21] In our model, $X = 1$ if $\theta = 1$, so in that state there is no risk in holding a. This assumption is not necessary for Lemma 3 as long as the cash flows in the high-state are sufficiently high.

sufficiently high. If this is not the case and shareholders are protected by limited liability or bank managers do not have substantial career concerns, they have an incentive to take on more risk than is socially efficient upon observing the stress test outcome S.[22] Designing stress test scenarios may, thus, be insufficient to elicit action from the bank's management and state-contingent capital regulation is necessary. The welfare benefits of the stress test are, thus, not only about reducing uncertainty about the asset values but also about aligning incentives in an informed way.

4 Optimal Sequential Stress Test

The optimal static stress test characterized by Proposition 1 is imperfectly informative about θ as to limit the loss of value of the bank's portfolio in the event of unfavorable news. It comes at the cost of having a significant rate of false-negative results in the stress test's pooling outcome. Importantly, the scope to set capital requirements only once coupled with the desire to make them contingent on the stress test outcome implies that the regulator must first share information about θ, and only then specify capital requirements. Such a "one-shot" view of capital regulation can be unnecessarily restrictive. For example, during the financial crisis of 2007-2009, the Federal Reserve and U.S. Treasury carried out a precautionary recapitalization of the major U.S. financial institutions via the Troubled Asset Relief Program (TARP) in late 2008. While not formally referred to as capital regulation,[23] it reduced the banks' riskiness by making them sell preferred equity to the Federal Reserve at fair market prices.[24] By stabilizing the banks in 2008, the Federal Reserve then implemented the highly informative stress test via the Supervisory Capital Assessment Program (SCAP) in early 2009, knowing that the banks could handle the stress test outcome even

[22]Even if the bank could be disciplined by the market, by, for example, the possibility of runs, it is a subtle question of whether the threat of not intervening is credible. On the other hand, maintaining a reputation for credible testing and setting capital requirements reinforces the regulator's reputation without risking systemic stability.

[23]Participation in TARP was de facto mandatory even for banks that were later found to be adequately capitalized by SCAP, e.g., NYT:"JPMorgan never needed the money, but was asked to take it — and complied for the sake of the weaker banks" (click for embedded link). As such, TARP could be viewed as a form of bank capital regulation.

[24]The Federal Reserve can provide loans to member banks through the "discount window" under the conditions specified in Section 10B of the Federal Reserve Act, which require these loans to be "be collateralized to the satisfaction of the lending Reserve Bank." Loans to individuals, partnerships, and corporations, such as primary dealers and insurance companies, are provided only under exigent circumstances and can be collateralized by sound valued privately issued collateral under Section 13.3 of the Federal Reserve Act or by government-issued securities under Section 13.13 of the Federal Reserve Act. In all cases, the collateral must "ensure protection for the taxpayer."

if it were unfavorable. Indeed, out of nineteen tested banks, ten were required to raise additional capital, which they succeeded in doing.

This section analyzes such multi-stage regulation under the framework of sequential stress tests and shows how sequential capital adjustments can mitigate the trade-off between risk-sharing and allocative efficiency. In the absence of trading frictions, we show that precautionary recapitalization (TARP), followed by a highly informative stress test (SCAP), constitutes an optimal sequential test. We also characterize the optimal sequential stress test under trading frictions and show that its expected payoff converges to the outcome of precautionary recapitalization if the expected asset riskiness is high, i.e., π_0 is close to π_{DF}.

4.1 Precautionary Recapitalization as an Optimal Sequential Stress Test

Definition 4 introduced *precautionary recapitalization* as a full or partial sale of the risky asset by the bank before any information about θ is revealed. A large precautionary sale can significantly reduce the bank's riskiness if $\theta = 0$ but also lead to a large asset misallocation if $\theta = 1$. The regulator can, however, partially address this inefficiency by fully disclosing θ once the asset is sold and relax the capital requirements in the event of $\theta = 1$, allowing the bank to reacquire the asset at the new price reflecting $\theta = 1$. In what follows, we show via martingale argument that precautionary recapitalization dominates any, potentially quite complex, sequential stress test.

A sequential stress test $\mathcal{S} = \left\{(S_n, R_n)_{n=1}^{N}\right\}$ specifies a (stochastic) posterior belief π_n about θ at the end of each step n, given by $\pi_n \overset{def}{=} \mathrm{E}\left[\theta \,|\, S_1, \ldots, S_n\right]$. Since any sequential test can be augmented by a final step that fully reveals θ without imposing additional capital requirements, it is without loss to impose that $\pi_N = \theta$. The sequence of signals and capital requirements also specifies the bank's portfolio (B_n, A_n) at the end of each step n determined inductively for $n \geq 1$ via (3) as

$$A_n \overset{def}{=} \frac{B_{n-1} + A_{n-1} \cdot \delta(1+\pi_n)/2 - d}{R_n \cdot \delta(1+\pi_n)/2}, \qquad B_n \overset{def}{=} d - \frac{1-R_n}{R_n} \cdot \Big(b + a \cdot \delta \frac{1+\pi_n}{2} - d\Big),$$

and where $(B_0, A_0) = (b, a)$ is the bank's starting portfolio. It is convenient to denote the corresponding (stochastic) market value of portfolio (B_n, A_n) by $W_n \overset{def}{=} B_n + A_n \cdot \delta \frac{1+\pi_n}{2}$. Following the optimality of allocating capital efficiently at the final step N of the stress test, the expected payoff at the end of step N of the sequential stress test $\mathcal{S}$ is given by $V(\pi_N, W_N)$, formally introduced by (6), which denotes the expected value of optimal asset allocation given belief π_N and wealth W_N. Given that $\pi_N = \theta$, the expected welfare from the sequential stress test $\mathcal{S}$ can be expressed as[25]

$$\begin{aligned}
\mathrm{E}\big[W_0+A_N X(1-\delta) - c\big(d-B_N-A_N X\big)\big] &= \pi_0 \cdot \mathrm{E}_1\left[V(1,\, W_N)\right] + (1-\pi_0)\cdot \mathrm{E}_0\left[V(0,\, W_N)\right] \\
&\overset{(i)}{\leq} \pi_0 \cdot V(1,\, \mathrm{E}_1\left[W_N\right]) + (1-\pi_0)\cdot V(0,\, \mathrm{E}_0\left[W_N\right]) \\
&\overset{(ii)}{=} \pi_0 \cdot V\big(1,\, \mathrm{E}_1\left[W_N\right]\big) + (1-\pi_0)\cdot V\left(0,\, \frac{W_0 - \pi_0 \mathrm{E}_1\left[W_N\right]}{1-\pi_0}\right) \\
&\overset{(iii)}{\leq} \pi_0 \cdot V(1,\, W_0) + (1-\pi_0)\cdot V(0,\, W_0). \qquad (7)
\end{aligned}$$

Inequality (i) follows from concavity of $V(\pi, w)$ in w, established in Lemma 2. Equality (ii) is a result of the self-financing property of the bank's portfolio implying that $\pi_0 \cdot \mathrm{E}_1\left[W_N\right] + (1-\pi_0)\cdot \mathrm{E}_0\left[W_N\right] = \mathrm{E}\left[W_N\right] = W_0$. Finally, inequality (iii) in (7) stems from the fact that the marginal value of wealth is lower in state $\theta = 1$ than in state $\theta = 0$ due to the possibility of distress costs. The expected wealth transfer from state $\theta = 1$ to state $\theta = 0$ is, however, limited by the fact that $\mathrm{E}_1\left[W_N\right] \geq W_0 \geq \mathrm{E}_0\left[W_N\right]$ as the value of the bank's portfolio, which is long the risky asset, grows (declines) in state $\theta = 1$ ($\theta = 0$) as the latter is being gradually revealed by the stress test.

Full precautionary recapitalization followed by a fully informative stress test achieves the upper bound in (7) of any sequential stress test by making the bank store all of its wealth W_0 in the safe asset and then disclosing θ perfectly. As the bank has no exposure to the risky asset after the initial sale, the market value of its portfolio is not affected by the disclosure, implying that its wealth is equal to W_0 in both states $\theta = 0$ and $\theta = 1$, thus attaining the upper bound in (7).

Proposition 2 (Precautionary Recapitalization)**.** *Suppose* $\pi_0 \geq \pi_{DF}$*. The optimal sequential stress test constitutes full precautionary recapitalization followed by a fully informative stress test. The bank is first required to sell all of its risky assets, then a stress test fully reveals* θ *and the regulator sets optimal capital requirements for each* $\theta \in \{0, 1\}$*.*

[25] For notational convenience, we write $\mathrm{E}_\theta\left[\cdot\right] = \mathrm{E}\left[\cdot|\theta\right]$.

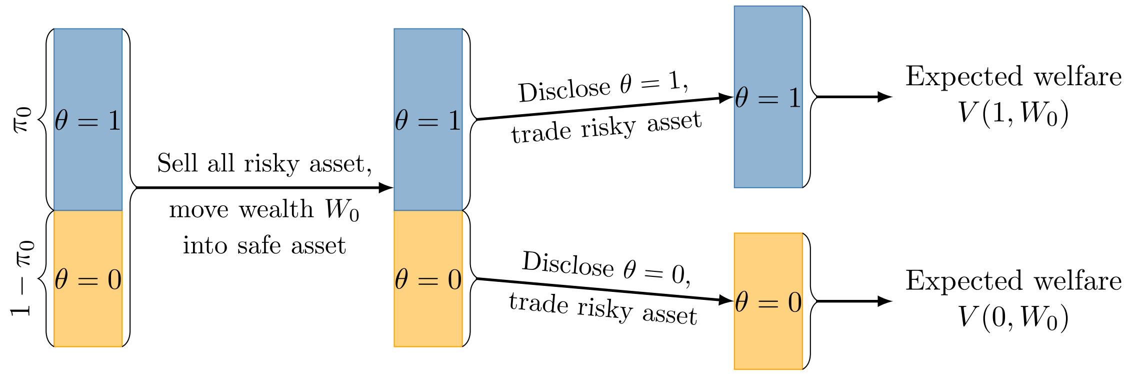


Figure 3: Timing and structure of optimal precautionary recapitalization.

The optimality of precautionary recapitalization in Proposition 2 stands in sharp contrast to the optimal static stress test in Proposition 1. The static stress test has to balance the benefits of risk-sharing with the costs of inefficient asset allocation. In contrast, precautionary recapitalization in Proposition 2 separates the problems of risk-sharing and allocation into two distinct steps, split by the irreversible disclosure of θ. The latter disclosure makes the bank's initial asset sale, effectively, irreversible, making it reacquire the asset in the second period at prices, which reflect the true fundamentals about θ and, crucially, without risk-sharing concerns, permitting ex-post efficient allocation. Even though the bank ends up with excess capital after precautionary recapitalization, the subsequent stress test is used to free up bank capital when it is not necessary. Surprisingly, this leads to an overall expected increase in the amount of capital the bank can use for asset purchases/investment and, thus, improves welfare. The irreversibility of information sharing about θ is similar to the irreversibility of time in Malenko and Tsoy (2019), but does not stem from agency conflicts. Instead, the optimality of sequential trade stems from the interaction between balance sheet adjustments and information disclosure and relies on the capital markets transacting at prices that reflect all available information at every point in time.

Precautionary recapitalizations are not just a theoretical construct and have been used during the financial crisis of 2007-2008 to stabilize the banks. The Troubled Asset Relief Program[26] (TARP) was implemented by the Federal Reserve and Treasury in the fall of 2008, not long before the

[26] As discussed in Veronesi and Zingales (2010), TARP was initially aimed at purchasing assets from the bank's balance sheets, but then pivoted to buying the banks' preferred equity. Our results hold if we allowed the banks to adjust their Capital Adequacy Ratios via equity issuances. Yet, the analysis of multiple banks in Sections 5 and 6 incorporates cross-bank trade and, focusing on the former, allows for a more self-contained analysis.

implementation of the Supervisory Capital Assessment Program (SCAP), the inaugural stress test, in early 2009. Having made the banks relatively safe, the regulator was then able to conduct a highly informative test without triggering unwanted distress. In this way, precautionary recapitalization is qualitatively different from an ex-post bailout. It is a form of ex-ante intervention, relying on ex-ante risk-sharing, rather than an ex-post capital infusion. Indeed, the convex distress cost $c(\cdot)$ in our model can be viewed as the shadow cost of bailout funds, and, as we saw in the optimal static stress test, the regulator may optimally avoid bailouts by implementing a partially informative test via an adverse scenario. This argument highlights another, subtle benefit of recapitalization prior to the stress test – it removes the need for the regulator to commit to a partially informative signal S, implemented as a fine-tuned adverse scenario, in the optimal static test. Instead, the policy involves full disclosure of θ once the bank is sufficiently well-capitalized.

Despite the benefits of precautionary recapitalization outlined earlier, the requirement for all banks to sell all of the risky assets for any level of belief and then allowing them to reacquire it in state $\theta = 1$ seems unrealistically burdensome. As such, it is best viewed as a theoretical benchmark highlighting the interaction between sequential balance sheet adjustments and information disclosure. Interestingly, the optimality of such extreme asset turnover holds despite the discount δ suffered by the bank when selling assets to the market, highlighting that it is not the initial underpricing that is important but, rather, the possibility of reacquiring these assets at that same discount δ in the future. In the next section, we aim to capture a more nuanced notion of transaction costs and study their implications on the optimal test.

4.2 Optimal Sequential Stress Test under Trading Frictions

Trading frictions may limit the bank's ability to reacquire the asset from the market, thus introducing costs to precautionary recapitalizations. Such market imperfections can be naturally captured in the model by endowing investors with bargaining power, which we model as different discounts δ_S and δ_B, such that $\delta_S \leq \delta \leq \delta_B \leq 1$, at which they are willing to buy or sell the risky asset respectively.[27] Investors are willing to pay $\delta_S \mathrm{E}[X]$ for the risky asset, below their autarky valuation

[27] Equivalently put, the bank sells the risky asset at discount $\delta_S \leq \delta$ and purchases it at discount $\delta_B \geq \delta$.

$\delta \mathrm{E}[X]$, and generating a positive expected return $(\delta - \delta_S)/\delta_S > 0$ for purchasing the asset. Similarly, investors are willing to sell the risky asset for $\delta_B \mathrm{E}[X]$, which is above their autarky payoff of $\delta \mathrm{E}[X]$, and generating a positive expected return $(\delta_B - \delta)/\delta > 0$ for selling the asset. From the perspective of the regulator, the bargaining power of investors does not itself make trading the asset inefficient since it is a zero-sum transfer from the banks and investors. The positive discount wedge $\delta_B > \delta_S$, however, drains the bank's balance sheet if it engages in round-trip transactions of first selling (at discount δ_S) and then buying (at discount δ_B) the asset.[28]

The advantage of a sequential stress test is best understood by considering the effect of a small, ε, precautionary recapitalization, prior to full disclosure of θ. This is costly for the bank if $\theta = 1$ as it ends up selling the asset at $\delta_S \frac{1+\pi_0}{2}$, which is below the price $\delta_B \mathrm{E}_1[X] = \delta_B$ of reacquiring it after $\theta = 1$ is disclosed. The costs of the strong bank stem from both a capital loss, captured by the ε sale carried out when the belief $\pi_0 < 1$, as well as market illiquidity, captured by the difference in discounts $\delta_S \leq \delta_B$. Such transactions drain the safe asset from the bank, which is socially costly as a unit of the safe asset in state $\theta = 1$ can be used to purchase $1/\delta_B$ units of the risky asset, for a total welfare gain of $(1-\delta)/\delta_B$. An expected social cost of an ε sale of the risky asset before θ is disclosed is the product of the ex-ante probability that $\theta = 1$, the magnitude of the capital loss, and the marginal social value of the safe asset in that state

$$\pi_0 \quad \times \quad \underbrace{\varepsilon\left(\delta_B - \delta_S \frac{1+\pi_0}{2}\right)}_{\textit{wealth loss if } \theta=1} \quad \times \quad \underbrace{\frac{1-\delta}{\delta_B}}_{\substack{\textit{marginal value} \\ \textit{of wealth if } \theta=1}} \qquad (\textit{presale cost})$$

To illustrate the potential benefits of a presale, denote by A the optimal portfolio of the bank in state $\theta = 0$ if the bank enters it with its starting portfolio of (b, a), obtained as the solution to[29]

[28] We introduce transaction costs as a percentage mark-up for buying and selling the asset. These discounts can be micro-founded by modeling a competitive market of small investors who pay a search cost to transact in the market for the risky asset. The banks compete via posted prices to attract investors and, thus, compensate them for the search costs, thus micro-founding the different discounts. Moreover, as the optimal stress test minimizes asset turnover to preserve bank balance sheet capacity, our results hold if we were to account for dead-weight costs to search by investors. While we model the bank improving its capital ratio via asset sales, similar round-trip financing costs are present if the bank were to raise capital via equity issuance both due to secondary market imperfections, but also income taxes faced by investors.

[29] If $\delta_S = \delta_B$, then $A = A(0)$ from Section 3.1. The safe asset holdings can be imputed from the budget constraint $B = b + [a - A]^+ \cdot \frac{\delta_S}{2} - [A - a]^+ \cdot \frac{\delta_B}{2}$.

$$A \overset{def}{=} \arg\max_{\hat{a}} \; \mathrm{E}_0 \left[\hat{a} \cdot (1-\delta)X - c\left(d - b - [a-\hat{a}]^+ \cdot \frac{\delta_S}{2} + [\hat{a}-a]^+ \cdot \frac{\delta_B}{2} - \hat{a} \cdot X \right) \right]. \tag{8}$$

The bank benefits most from precautionary recapitalization in state $\theta = 0$ if the optimal portfolio A does not prescribe to reacquire the asset at the higher discount, i.e., if $A < a$. In this case, an ε asset presale leads to a net wealth gain of $\varepsilon \cdot \delta_S \left((1+\pi_0)/2 - 1/2\right) = \varepsilon \cdot \delta_S \pi_0/2$. The expected social benefit of the precautionary recapitalization is the product of the ex-ante probability of $\theta = 0$, the capital gain obtained if $\theta = 0$, and the marginal value of safe asset holdings in state $\theta = 0$, obtained by differentiating (8) with respect to b and applying the Envelope theorem,

$$(1-\pi_0) \times \underbrace{\varepsilon \delta_S \frac{\pi_0}{2}}_{\substack{\text{wealth gain} \\ \text{if } \theta=0}} \times \underbrace{\mathrm{E}_0 \left[c'\!\left(d - b - [a-A]^+ \cdot \frac{\delta_S}{2} + [A-a]^+ \cdot \frac{\delta_B}{2} - A \cdot X \right) \right]}_{\geq (1-\delta)/\delta_B, \quad \text{marginal social value of wealth if } \theta=0}. \qquad (\textit{presale benefit})$$

The marginal value of wealth in state 0 is always weakly higher than $(1-\delta)/\delta_B$ – the per-dollar social gain of purchasing more of the risky asset – and, moreover, strictly exceeds it whenever distress occurs with positive probability and convex costs of distress come into play. Comparing the marginal benefit of a presale to its cost, we see that precautionary recapitalization is beneficial if the reduction in distress costs in state $\theta = 0$ exceeds the expected misallocation in state $\theta = 1$:

$$\underbrace{\mathrm{E}_0 \left[c'\!\left(d - b - [a-A]^+ \cdot \frac{\delta_S}{2} + [A-a]^+ \cdot \frac{\delta_B}{2} - A \cdot X \right) \right]}_{\geq (1-\delta)/\delta_B} \geq \frac{1-\delta}{\delta_B} + \frac{1-\delta}{1-\pi_0} \cdot \left(\frac{2}{\delta_S} - \frac{2}{\delta_B} \right). \tag{9}$$

Inequality (9) is always satisfied if $\delta_S = \delta_B$ as, in the absence of trading frictions, the marginal value of wealth is weakly higher in state $\theta = 0$ than in state $\theta = 1$, and offers an alternative intuition for the optimality of full precautionary recapitalization in Proposition 2. If, however, $\delta_S < \delta_B$, then trading frictions impose a cost on precautionary recapitalization, especially if state $\theta = 0$ is unlikely: if π_0 is close to 1, then (9) is never satisfied as the r.h.s. becomes exceedingly large. Precautionary recapitalization can, thus, be valuable only if the ex-ante riskiness $1 - \pi_0$ is sufficiently high, but even then, is limited by the point at which the marginal value of wealth in state $\theta = 0$ falls below the transaction-adjusted costs in state $\theta = 1$.

Lemma 4 (Precautionary Recapitalization under Frictions)**.** *Suppose the marginal cost of distress* *is sufficiently large, i.e.,* $c'(0) \geq \bar{c}$*, and* $\pi_0 \geq \pi_{DF}$*. It is optimal to precautionary recapitalize the bank and fully reveal* θ *only if the probability of distress is sufficiently high, i.e.,* $\pi_0 \leq \bar{\pi}$*, where* $\bar{\pi} < 1$ *if* $\delta_S < \delta_B$*. Moreover, for* $\pi < \bar{\pi}$ *the optimal precautionary recapitalization requires the bank to sell only the necessary fraction* $\frac{d-b-a\cdot\delta_S/2}{\delta_S\cdot\pi_0/2}$ *of its risky asset as to avoid distress in state* $\theta = 0$*.*

In the presence of transaction costs, the magnitude of the optimal precautionary recapitalization reflects the expected riskiness of the bank. Once it is conducted, the bank is required to raise additional capital if and only if it fails the stress test, similar to the SCAP outcome in 2009 when ten out nineteen banks failed the test and were required to improve raise additional capital. A sequential test provides an additional degree of adjustment to the regulator by letting her acquire and disclose additional information about θ prior to engaging in precautionary recapitalization characterized in Lemma 4. We show that, in contrast to Proposition 2, the regulator discloses information about θ before requiring the bank to raise capital - the optimal stress test features both sequential communication and sequential recapitalization. First, the regulator acquires a partially informative signal about θ, similar to the adverse scenario static test in Proposition 1. Then, she imposes precautionary recapitalization *only* if the bank fails the first test and the posterior belief about asset riskiness is sufficiently high, thus avoiding recapitalization if the strong bank passes the first test. Finally, the regulator fully discloses θ and lets the $\theta = 1$ banks who failed the first test to reacquire some of their sold asset. Similar to Proposition 2, the optimal sequential stress test can be implemented in two steps, as depicted in Figure 4.

Proposition 3 (Optimal Sequential Stress Test)**.** *Suppose* $c'(0) \geq \hat{c}$*,* $\pi_0 \geq \pi_{DF}$*, and* $\delta_B \leq \delta \leq \delta_S$ *with at least one of the inequalities being strict. Then the optimal sequential stress test has two steps. The first test is a pass/fail adverse stress test with* $\mathrm{P}(\theta = 1 \,|\, S = \textit{fail}) = \pi^{**} \leq \pi_{DF}$*. If the bank fails the first test, it undergoes recapitalization and, after that, a second, fully informative, stress test is conducted.*

The first stress test sometimes passes the bank if the state is $\theta = 1$ and fails otherwise. The optimal rate of false negatives ensures that the market value of the bank's portfolio upon failing the stress test is still sufficiently high to mitigate distress costs by selling the risky asset to the market.[30]

[30] When $c'(0)$ is sufficiently large, then $\pi^{**} = \pi_{DF}$ and the bank is risk-free even if $\theta = 0$. However, for intermediate

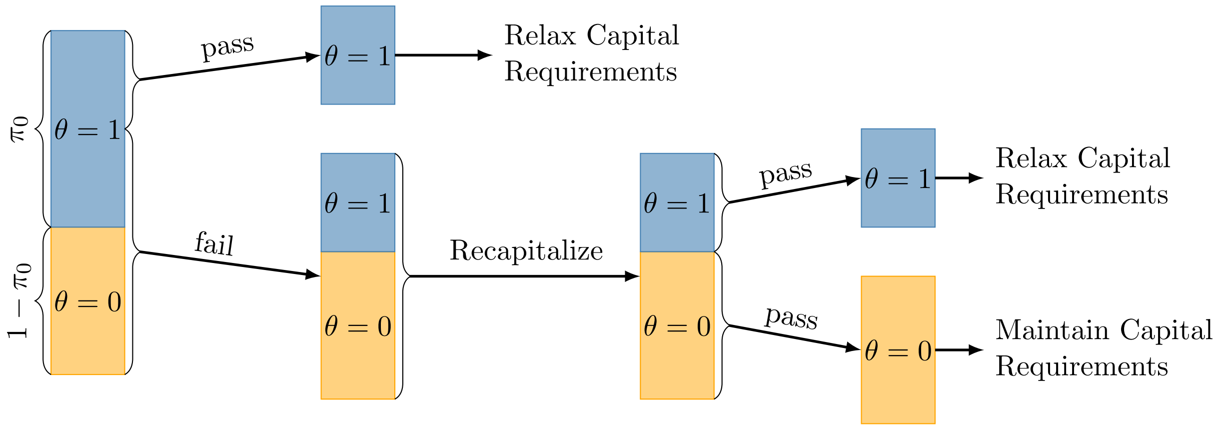


Figure 4: Optimal sequential stress test begins with an adverse pass/fail signal followed by optimal precautionary recapitalization in the event the bank fails it, followed by a fully informative test.

Upon failing the first stress test, the bank is required to improve its capital ratio to $R_1 = 1$, and it does so by selling its risky asset. Upon recapitalization following a failing test, the bank holds d units in safe assets and no risky asset. If the bank passes the first test, then it is found adequately capitalized, and it can increase its asset holdings. After the first stress test and recapitalization, the regulator fully reveals θ. The second stress test aims not to improve the stability of the bank further but, instead, to undo the negative effects of the initial recapitalization in case of a false negative outcome for the $\theta = 1$ bank. If the second stress test reveals that $\theta = 0$, the capital requirements continue to be tight to mitigate the risk of distress. However, if the regulator discloses that $\theta = 1$ and the asset is not risky, it can allow the bank to purchase back some of the risky assets.

If the marginal cost of distress is sufficiently high, then the first test of the optimal sequential stress test relies on an adverse scenario, failing some of the $\theta = 1$ banks to preserve the value of the bank's balance sheet. The intuition is similar to Proposition 1, but the severity of the optimal scenario may differ.[31] Once the bank recapitalizes at the first step of the test, the regulator implements a less adverse scenario for the failed banks. Given the second scenario's less adverse nature, the strong bank passes it with certainty, leading to the full revelation of the underlying state θ.

values of marginal distress cost $c'(0)$ the optimal sequential test described by Proposition 3 entails $\pi^{**} < \pi_{DF}$, corresponding to the bank entering distress with a positive probability if $\theta = 0$.

[31]If $c'(0)$ is sufficiently large then both static and sequential stress tests are default-free and the optimal adverse scenario is the same in both.

Solution approach. The optimal sequential stress test in Proposition 3 is a dynamic mechanism in which the regulator discloses information about θ to both the bank and investors and specifies portfolio allocations. There is no standard approach to characterizing such mechanisms, especially in a model aimed at capturing the applied nature of the stress testing exercise. We achieve traction by considering a relaxed version of the problem in which conditional on $\theta = 1$, the bank faces no liquidity frictions and can increase its balance sheet by taking on additional debt. This allows for tractable characterization of the optimal two-period stress test for any starting portfolio (b, a). We then show that the welfare objective cannot be improved via additional signals or trades prior to the first test. This is sufficient to apply a backward induction argument to prove the optimality of the two-step stress test in the relaxed problem. We then show that the relaxed problem's policy functions attain the same payoff in the original problem.

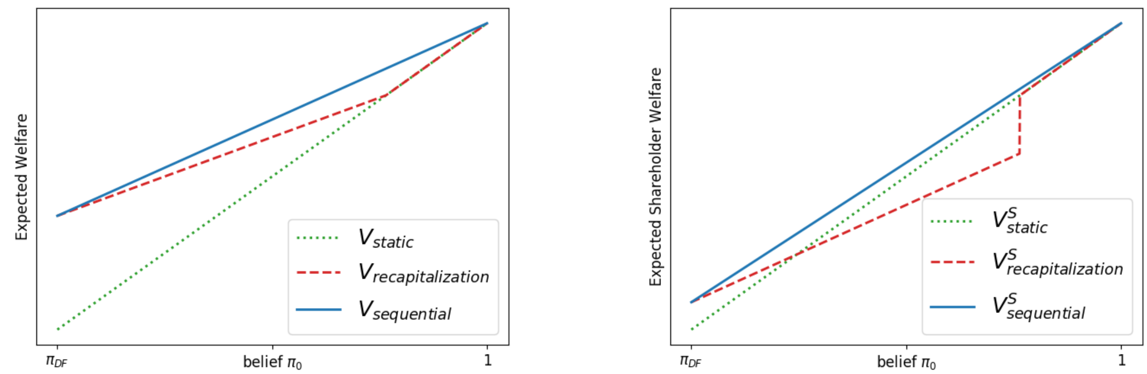
4.3 Comparison of Optimal Policies

The characterization of the optimal sequential stress test in Proposition 3 allows us to evaluate the performance of simpler policies, such as the optimal static test in Proposition 1 and optimal precautionary recapitalization in Lemma 4.

Figure 5 evaluates the three policies as a function of initial belief π_0 about θ. Figure 5a shows that precautionary recapitalization does just as well as the sequential test when beliefs are low, i.e., when the bank's starting capital adequacy ratio is low. They both do substantially better than the static test as they avoid asset misallocation in state $\theta = 1$. The sequential test highlights the value in the interaction between the first-step stress test and the initially imposed capital requirements and provides the greatest gains for intermediate beliefs π_0.

When the marginal distress cost $c'(0)$ is sufficiently large, all policies implement a default-free outcome for the bank shareholders whenever $\pi_0 \geq \pi_{DF}$.[32]. As we see in Figure 5b, the expected welfare to shareholders is steeper under the optimal static test, than under the optimal sequential test. This arises due to the punishment imposed by the static stress test on $\theta = 1$ banks who are failed by the test due to the optimal adversity of the static stress test scenario. If the bank were to,

[32]This is not the case for precautionary recapitalization which leads to the bank's distress in state $\theta = 0$.

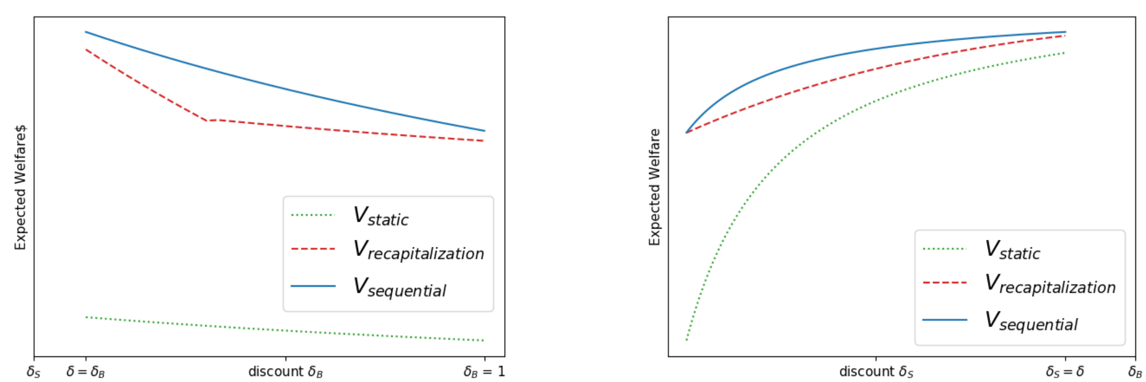


(a) Expected social welfare as a function of starting beliefs π_0.

(b) Expected shareholder welfare as a function of starting beliefs π_0 under $c_S(x) = 0$.

Figure 5: Expected social and shareholder welfare under optimal static test, precautionary recapitalization, and sequential test as functions of beliefs $\pi \in [\pi_{DF}, 1]$. Parameters: $b = 0.1$, $a = 1$, $d = 0.65$, $\delta = 0.8$, $\delta_S = 0.7$, $\delta_B = 0.9$, $c(x) = 4 \cdot \max(0, x)$.

say, make ex-ante costly investments into the quality of its portfolio, then it has sharper incentives to invest in the case of the static test due to this additional punishment.



(a) Expected social welfare as a function of δ_B, given $\delta_S = 0.7$.

(b) Expected social welfare as a function of δ_S, given $\delta_B = 0.85$.

Figure 6: Expected social welfare under optimal static test, recapitalization, and sequential test as functions of discounts δ_S and δ_B. Parameters: $\pi_0 = 0.8$, $b = 0.1$, $a = 1$, $d = 0.65$, $\delta = 0.73$, $c(x) = 4 \cdot \max(0, x)$.

Figure 6 illustrates the social welfare achieved by the optimal policies as a function of the trading frictions. Figure 6a plots the expected welfare under the optimal sequential and static tests as one increases the price at which the bank buys back the assets from the market, captured by increasing δ_B from δ to 1, all while maintaining a constant discount $\delta_S < \delta$. A change in δ_B does not change the stress tests' informativeness if $c'(0)$ is sufficiently large so that the stress tests are default free.

However a greater δ_B makes it more costly for the bank to buy the risky asset from the market. The first stage of the optimal sequential and optimal static tests are equally affected by the increase in δ_B. However, the reduced ability to purchase risky assets back after a large recapitalization further reduces welfare under the sequential test. The sequential stress test outperforms the static one even if $\delta_B = 1$ as the expected welfare is still improved by the more efficient allocation of the risky asset in state $\theta = 1$, despite this benefit being extracted by investors and not captured by the bank.[33]

The bank's solvency crucially depends on the market value of its assets, captured by the discount δ_S applied to the bank if it tries to sell them. Figure 6b shows that as δ_S decreases, while keeping a constant discount $\delta_B > \delta$, the relative value of conducting the sequential stress test over the static one increases. A lower sale price of the risky assets lowers the maximum amount of cash that the bank can raise via asset sales and lowers its risk-weighted capital ratio. As a result, the optimal static and the first stage of the optimal sequential tests reveal less information to sustain higher asset prices conditional on failing the stress test. A higher rate of false negatives lowers social welfare because it forces the bank to undergo costly recapitalization more frequently, even when $\theta = 1$. The relative value of conducting the sequential stress test is higher if δ_S is lower since it allows the regulator to undo the negative effects of increasingly uninformative recapitalization. In terms of stress test scenarios, a lower δ_S results in a more adverse and less informative stress test scenario, making it more valuable to disclose additional information about θ to the market.

4.4 Regulatory Commitment to Stress Test Disclosures

The literature on stress test disclosure, surveyed by Goldstein and Leitner (2020), has long emphasized the importance of regulatory commitment to partial transparency.[34] Nevertheless, central bankers are often hesitant in accepting the benefits of incomplete information acquisition and disclosure, guided by their desire to have a clear understanding of bank risks while, at the same time, being worried about both how the market will interpret withholding of information. In this context, the fact that precautionary recapitalization followed by a fully informative stress test can approxi-

[33] If $\delta_B > 1$, then bank shareholders prefer not to purchase the asset as there are no longer gains from trade.

[34] There are multiple ways to justify such commitment power: stylized scenarios introduced in Section 2.1, public information disclosure akin to Dye (1985), or regulator's long-run reputation studied in Mathevet, Pearce, and Stacchetti (2019).

mate the globally optimal sequential stress test and, at the same time, alleviate the need to commit to partial transparency altogether presents the regulators with a useful policy tool. Such a policy requires only that the regulator shares information once and fully. Moreover, post recapitalization, it is in the best interest of the regulator to both acquire as much information as possible, as well as report it truthfully – misreporting a $\theta = 0$ as a $\theta = 1$ state would lead the bank to acquire lots of risky assets at high prices, which would set the regulator up for high distress costs. Precautionary recapitalization is, thus, a substitute for the opacity necessary under the optimal static test.

An interesting dimension that our analysis does not capture is the possibility of the regulator being privately informed about bank's riskiness, similar to the informed principal problem of Maskin and Tirole (1992). If the regulator knew θ perfectly prior to designing a stress test, then the unique equilibrium is a perfectly separating one due to the opposing incentives of the two types. The regulator who knows that $\theta = 1$ would like the bank to expand its balance sheet. This action is too costly for the $\theta = 0$ type to mimic due to the high expected distress cost. However, when the regulator's private information gives rise to interior beliefs, then the more pessimistic regulator might have an incentive to mimic a more optimistic regulator by running a stress test with a lower rate of false negatives. Such deviation would signal higher asset prices and reduce the necessary bank failure rate. Moreover, mimicking the optimistic regulator comes at a low cost if the stress test implements a default-free allocation. If, however, the optimistic regulator's optimal test induces distress, as is the case whenever $\pi^{**} < \pi_{DF}$, then the pessimistic regulator has an incentive to reveal itself as it faces greater exposure to bank distress were it to mimic the optimistic regulator.

5 Stress-Testing Aggregate versus Idiosyncratic Risk

The analysis of Sections 3 and 4 considered the optimal stress tests in a model with homogeneous banks. That allowed us to identify a fundamental economic trade-off: stress tests can be used to increase the banking sector's safety at the cost of changing the allocation of assets. We showed how sequential stress tests could help achieve desired safety at the lowest re-allocation cost.

In this section, we start by noticing that, in practice, regulators deal with heterogeneous banks. On the one hand, it makes the regulator's job harder when the optimal stress test would vary with each bank's position. On the other hand, it creates the opportunity to re-allocate some of the risky assets within the banking sector, which presumably is less inefficient than making the banks sell the assets to outside investors. We show that sequential tests increase the banking system's aggregate risk-bearing capacity by re-allocating the risky assets efficiently across banks. An optimal (sequential) stress test in such an environment starts with some interbank trade, followed by offloading some aggregate risk to the outside capital investors. Precautionary recapitalization prior to the stress test turns out to be beneficial when the banking liquidity is low.

When stress testing a financial system comprised of heterogeneous banks, the regulator must consider the distinct implications of aggregate and idiosyncratic risk. Aggregate risk is the uncertainty over the total quantity of risk across banks, while idiosyncratic risk is the uncertainty over which of the many banks hold this risk. If all banks hold low quality risky asset, then, to reduce their risk exposure, they must sell some of the assets to outside investors. If, however, only some banks hold low quality risky asset, then it may be possible to sell some of this asset to other banks, not outside investors. The regulator, now, faces a new trade-off: a more adverse stress test improves the average quality of the failing banks, facilitating their recapitalization, but reduces the number, and, hence, the total liquidity available to the passing banks, limiting their overall capacity to buy the asset from the failing banks and forcing asset sales to outside investors.

Consider an extension of our model in which there is a unit mass of banks indexed by $j \in [0,1]$. Each bank j holds a units of the risky asset with cash flow $X_j \in \{1,\ X\}$, where $X \sim U[0,1]$. The aggregate state $\theta \in \{0,1\}$ pins down the total quantity of risk held by the banks by specifying a fraction μ_θ of (strong) banks who have the high-quality asset $X_j = 1$ and a fraction $1-\mu_\theta$ of (weak) banks who hold the low-quality asset $X_j = X$.[35] State $\theta = 1$ denotes low aggregate risk, captured by $\mu_1 \geq \mu_0$. Idiosyncratic risk is captured by the uncertainty over which of the banks hold the bad asset conditional on μ_θ. Without loss, we assume there are no high-quality assets outstanding in the market ex-ante, while there is a perfectly elastic supply of the low-quality assets $X \sim U[0,1]$.

[35]The single-bank model corresponds to the case of pure aggregate risk given by parameters $1 = \mu_1 > \mu_0 = 0$ in which all banks are identical, and the portfolio of each one is pinned down by the realization of aggregate state θ.

To minimize new notation, we make two simplifying assumptions governing interbank trade, neither of which have qualitative implications on the results. First, we assume that the bank which sells the risky asset in the interbank market does not have bargaining power, pinning down the sale price of the asset to be equal to $\delta_S \cdot \mathrm{E}[X]$, capturing the bargaining advantage of the acquiring bank, for which the capital adequacy constraint is satisfied even if it does not trade. Second, motivated by the banks' superior ability to monitor and manage the risky asset, we assume that each unit of the risky asset pays X if it is held by one of the banks, independent of whether the retaining bank is the asset's originator or not. Finally, for tractability (to manage the problem's increased dimensionality when working with multiple banks), we focus on default-free stress tests which ensure that no bank enters distress in period $t = 2$.[36]

5.1 Pure Idiosyncratic Risk

First, consider the case of pure idiosyncratic risk, as captured by $\mu_1 = \mu_0 = \mu$. Bank liquidity, as measured by safe asset holdings b, provides a medium for reallocation of risk in the interbank market. The optimal static stress test passes the maximum number of strong banks subject to keeping asset prices for the failing banks sufficiently high so that they can recapitalize safely. In the presence of an interbank market, the passing banks can use their safe assets to purchase the risky assets of the failing banks, leading to an unambiguous improvement to social welfare over an economy without interbank trade. When the failing banks need to raise more capital than the liquidity available to the passing banks, however, the optimal static stress test is unable to reallocate *all* of the failing banks' risky assets within the banking system and resorts to inefficient sales to outside investors. Precautionary recapitalization improves upon the static test whenever the latter resorts to such outside sales. The optimal sequential stress test generates further improvement by tapping into the liquidity of *all* strong banks, which achieves maximum risk-sharing across banks.

Proposition 4 (Idiosyncratic Risk). *Suppose there is no aggregate risk, i.e., $\mu_1 = \mu_0 = \mu \geq \pi_{DF}$, and strong banks have sufficient balance-sheet capacity, i.e., $b + a \cdot \mu \geq d$. The* ***optimal default-free***

[36]This simplifying assumption is motivated by the fact that the optimal static and sequential stress tests for a single bank derived in Sections 3 and 4 are default-free as long as $c'(0)$ is large enough.

– ***static stress test*** *is an adverse pass/fail test with* $\mathrm{P}(X_j = 1|S_j = \text{fail}) = \pi_{DF}$. *Moreover, the banks avoid selling risky assets to outside investors if and only if* b *is sufficiently large, i.e.,*

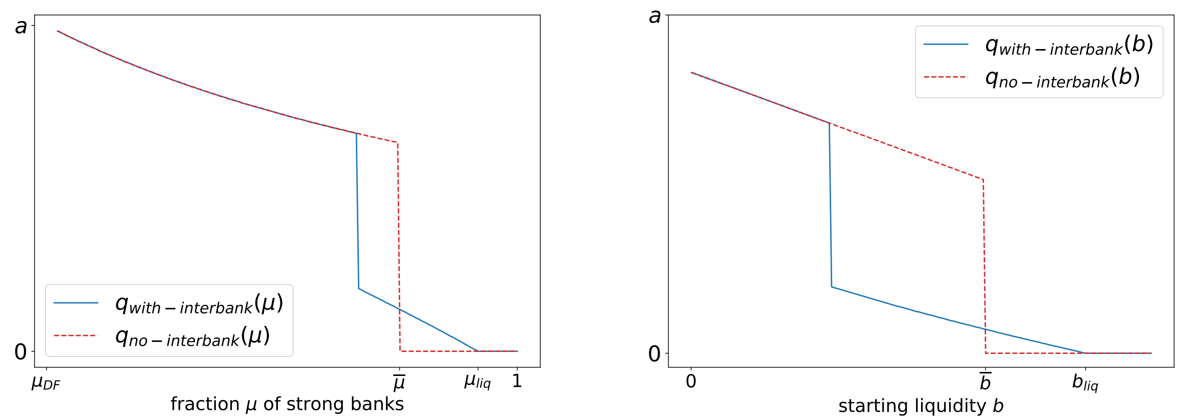
$$\frac{b + a \cdot \delta_S(1+\mu)/2 - d}{b + a \cdot \delta_S - d} \geq \frac{d-b}{d}. \tag{10}$$

– ***precautionary recapitalization*** *is always valuable if the passing banks do not have sufficient liquidity to purchase the risky asset from the failing banks, i.e., if condition* (10) *is not met.*

– ***sequential stress test*** *reallocates all of the risky asset across the banks and allows the banking system to purchase more risky assets from the market.*

Just like in case of the optimal static stress test for a single bank in Proposition 1, the optimal test for a cross-section of banks relies on subjecting every bank to the same adverse scenario, failing each weak bank with certainty, and failing each strong bank with probability $f = f_{DF} = \frac{1-\mu}{\mu(1-\pi_{DF})}$. The rate of false negatives f is just high enough to allow the weak banks to recapitalize and avoid distress at $t = 2$. A total of $\mu \cdot (1 - f_{DF})$ strong banks pass the test and step in as natural buyers of the assets sold by the banks that failed the test. The strong passing banks can acquire at most $\mu \cdot (1 - f_{DF})b$ worth of assets while the failing banks need to raise $\big(\mu \cdot f_{DF} + 1 - \mu\big) \cdot \big(d - b\big)$ to avoid distress, which gives rise to condition (10). Whenever the passing banks lack the liquidity to purchase all of the assets sold by the banks that fail the test, the remaining assets are sold to outside investors imposing a welfare loss. While the resulting allocation differs from the case of a single bank optimal stress test in Proposition 1 by virtue of interbank trade, the form of the optimal static stress test and prescribed capital requirements for each bank are identical.

The optimal static test does not fully take advantage of the cross-bank risk-sharing since failing some strong banks to support asset prices also takes up the interbank market's liquidity. Precautionary recapitalization prior to a stress test has two effects. First, it increases the safe assets held by each bank at the cost of selling some of the high quality risky assets to the outside market. This step turns out to be welfare-neutral as, in the absence of aggregate risks, banks would have to sell the asset to outside investors with certainty anyway. Second, it increases the informativeness of the subsequent stress test (lower f), passing more strong banks and unlocking their liquidity to purchase the assets of the failing banks. Precautionary recapitalization, thus, improves welfare by exploiting the complementarity between a transparent stress test and interbank trade.



(a) Optimal recapitalization comparison as a function of fraction μ of strong banks; $b = 0.05$

(b) Optimal recapitalization comparison as a function of bank's starting liquidity b; $\mu = 0.86$.

Figure 7: Solid blue line is optimal precautionary recapitalization if banks can trade the risky asset among themselves. Red dashed line is optimal precautionary recapitalization if banks can trade only with the capital market. Parameters: $a = 1$, $d = 0.6$, $\delta = 0.8$, $\delta_S = 0.7$, $\delta_B = 0.9$.

To highlight the subtle implications of interbank trade on the optimal stress test, Figure 7 compares optimal precautionary recapitalization policies with and without interbank trade.[37] Figure 7a shows that, in the absence of interbank trade, there exists a belief threshold $\bar{\mu}$ such that for $\mu \geq \bar{\mu}$ the banking sector is sufficiently safe so that the liquidity cost of precautionary recapitalization imposed on the $\theta = 1$ banks does not justify the benefit of a more informative stress test. If, however, the strong banks can utilize their liquidity to purchase assets from the failing banks on the interbank market, a more informative stress test allows the banking system as a whole to reduce the total amount of risky assets sold to the capital market. This force reduces the social cost of precautionary recapitalization and makes it optimal to recapitalize the banks for $\mu \in [\bar{\mu}, \mu_{liq}]$ with the prospect of using this liquidity in the interbank market. For $\mu < \bar{\mu}$, however, interbank markets increase the opportunity cost to precautionary recapitalization. Any asset sale beyond what is necessary for the passing banks to purchase the assets of the failing banks reduces the quantity of assets of the failing banks and forces the passing banks to purchase them from the capital market at the higher discount $\delta_B > \delta_S$, thus draining their balance sheet. Thus, the optimal magnitude of

[37]See Online Appendix B.3 for details on the numerical implementation.

precautionary recapitalization is lower in the presence of an interbank market when the banking sector riskiness increases. Figure 7b illustrates similar non-monotonic effect of interbank trade on optimal recapitalization with respect to bank liquidity b.

A sequential stress test consisting of multiple steps of information disclosure and capital adjustments allows the regulator to make use of all liquidity available to the strong banks and reallocate the risky assets only within the financial system whenever strong banks have enough liquidity, formally stated as $b+a\cdot\mu \geq d$. For some parameters[38] the optimal sequential stress test can be implemented in just two steps. During the first step, the stress test fails strong banks with probability $\hat{f} > f_{DF}$. The failing banks undergo partial recapitalization by selling a fraction of their risky assets to $\mu\hat{f}$ strong banks that pass the first stress test. The high false-negative rate $\hat{f} > f_{DF}$ allows the regulator to then conduct a fully informative test in second step. At this point, only weak banks fail the test and are required to further improve their capital ratios by selling more risky assets, which can be purchased by all the strong banks which pass the second, fully informative stress test.

5.2 Pure Idiosyncratic Risk with Observed Heterogeneity

In practice, banks may also differ along publicly observable dimensions such as the safe asset holdings b, risky asset holdings a, or liabilities d. To explore the effects of observable heterogeneity, suppose, without loss,[39] that there are two groups of banks $j \in [0,1)$ and $j \in [1,2)$ that, in addition to unobservable riskiness of portfolios $X_j \in \{X, 1\}$, also publicly differ in the quantity of the safe assets, specifically $b_j = b_L$ for $j \in [0,1)$ and $b_j = b_H$ for $j \in [1,2)$ with $b_H > b_L$. As shown in Proposition 4 and Lemma 3, the optimal default-free test S^L for the $b_j = b_L$ bank is pass-fail and is less informative than the optimal test S^H, which is also pass-fail, for the $b_j = b_H$, captured by

$$\begin{aligned}\mathrm{P}(\theta = 1 \mid S_j = \mathit{fail}, b_j = b_L) = \pi_{DF,L} &\overset{def}{=} \frac{2}{\delta_S} \cdot \frac{d - b_L}{a} - 1 \\ &> \frac{2}{\delta_S} \cdot \frac{d - b_H}{a} - 1 \overset{def}{=} \pi_{DF,H} = \mathrm{P}(\theta = 1 \mid S_j = \mathit{fail}, b_j = b_H)\,.\end{aligned}$$

[38] For example, if $a = 1$, $d = 0.5$, $\delta_S = 0.5$, and $\mu = 0.5$, then constraint (10) requires $b \geq 0.298$ while a two-step sequential stress test generates an efficient outcome for $b \geq 0.183$. See condition (A.76) in Lemma A.15 in Appendix A for a sufficient condition for the two-step sequential test that reveals the strong banks in two steps is optimal.

[39] We focus on heterogeneity only in safe asset holdings b to minimize new notation, but the argument applies if the public heterogeneity is in any of these, even combined, dimensions, including ex-ante priors of asset quality.

If the regulator could run *different* stress test scenarios for the two groups of banks, then she would subject poorly capitalized $b_j = b_L$ banks to the more adverse scenario, corresponding to signal S^L, to reduce the stigma of failing the test, while subject the better capitalized $b_j = b_H$ banks to a less adverse scenario, corresponding to signal S^H. Bank regulators, however, are often hesitant in allowing for heterogeneous stress scenarios as it may give the banks an incentives to conceal losses, as the poorly capitalized b_L shareholders would, generally, benefit from a less adverse scenario at the expense of social distress costs. If, however, the stress test scenario had to be *the same* for the whole system, then we show that the stress test informativeness has to cater to the poorly capitalized $b_j = b_L$ banks in order to keep them safe, thus distorting allocations for the well capitalized b_H banks. Such an externality creates an additional benefit for sequential stress tests.

Corollary 3 (Observed Heterogeneity)**.** *Suppose the regulator must subject all banks to the same scenarios. Then, the* ***optimal static test*** *for the cross-section is given by* S^L*, and implements the optimal static test for the* $b_j = b_L$ *banks. The well-capitalized* $b_j = b_H$ *banks are allowed to retain more risky assets upon failing this test than the* $b_j = b_L$ *banks, but fail the test more frequently than their individually optimal test. The* ***optimal sequential stress test*** *achieves greater welfare than the sum of the individually optimal stress tests by implementing the allocation of the representative bank, while subjecting both banks to the same stress scenarios at each step.*

Since the optimal static stress test S^L for $b = b_L$ banks is less informative, it fails too many of the weak $X_j = X$ but well-capitalized $b_j = b_H$ banks. Such a high failing rate forces the failing $b_j = b_H$ banks to hold an excessive amount of capital at the high belief $\pi_{DF,L} > \pi_{DF,H}$. Moreover, the increased number of such failing banks reduces their purchasing capacity in the interbank market, further reducing efficiency. The optimal sequential stress test utilizes the fact that S^H is strictly more informative than S^L, thus running the latter scenario first, and subsequently refining the signal partition. While all banks are subjected to the same scenario in this case, the sequential testing allows the banks to recapitalize at different steps of the test.

5.3 Combining Aggregate and Idiosyncratic Risks

The optimal stress test in the presence of both aggregate and idiosyncratic risks combines the intuitions from the pure aggregate risk case, analyzed in Sections 3 and 4, with the pure idiosyncratic risk case, analyzed in Section 5.1. We describe the optimal tests below, focusing on ex-ante symmetric banks' case, and, to save space, relegate the formal results to Appendix B.1.

The optimal **static stress test** generates an inverse relation between idiosyncratic risk sharing in the interbank market and aggregate risk sharing between the banks and the capital market. It reveals low aggregate risk, $\mathrm{P}(\theta = 1|S) = 1$, with probability $\frac{\pi - \bar{\pi}}{1-\bar{\pi}}$.[40] Once it is known that $\theta = 1$, the regulator simply conducts the optimal static test of Proposition 4 and reveals some bank specific information to facilitate interbank trade. When the aggregate stress test outcome is mixed, i.e., $\mathrm{P}(\theta = 1|S) = \bar{\pi} \geq \pi_{DF}$, the regulator can reveal only μ_0 of strong banks. The average quality of the banks that fail the stress test is $\bar{\pi} \geq \pi_{DF}$ which facilitates recapitalization of the weak banks but reduces the number of strong banks that pass the stress test (μ_0), limiting the interbank liquidity and dampening cross-bank risk sharing. This highlights the role of macro-prudential stress tests in moving risk outside the core of the financial system, but only if it is constrained on an aggregate level. The stress test limits disclosure of weak banks when expected aggregate risk is high, highlighting sub-optimality of revealing bank-specific information. In other words, our results discover the following asymmetry: the optimal test reveals more information about individual banks when the aggregate risk is low and less bank-specific information when the aggregate risk is high.

Precautionary recapitalization not only improves risk-sharing across states, as discussed in Section 4.1 but also increases informativeness of the subsequent stress test, consequently improving interbank risk-sharing. Precautionary recapitalization always improves welfare relative to the optimal static stress test when condition (10) does not hold for $\mu = \mu_1$, that is when the banking system is unable to avoid risky asset sales to outside investors even if the aggregate risk is low. In this case, precautionary recapitalization improves interbank risk-sharing regardless of the aggregate level of risk. When (10) holds for $\mu = \mu_1$ then precautionary recapitalization decreases welfare if

[40] Threshold $\bar{\pi}$ is pinned down by the fact, that conditional on $\mathrm{P}(\theta = 1) = \bar{\pi}$ the regulator optimally conducts a stress test that exactly μ_0 strong banks pass and all other banks fail.

aggregate risk is low, but is still beneficial upon $\mathrm{P}(\theta = 1|S) = \bar{\pi}$. The second effect dominates whenever the expected amount of aggregate risk is high, i.e., π_0 is low and close to $\bar{\pi}$.

The optimal **sequential stress test** improves upon the precautionary recapitalization outcome by allowing the regulator to first fully disclose the level of aggregate risk θ and then reallocate risky assets across the banks via the optimal sequential stress test from Proposition 4. Effectively, the optimal sequential stress test fully disentangles the revelation of aggregate risk from idiosyncratic risk by inducing a high degree of interbank risk-sharing. A consequence of this is that the optimal sequential stress test resorts to cross-state risk-sharing only when the opportunities for the reallocation of risk within the financial system are fully exhausted.

In presence of observable heterogeneity among the bank's balance sheets, the correlated nature of the aggregate risk imposes information spillovers across banks *even* if the regulator could subject banks to different stress scenarios in a static test. In this case, the sequential improves upon the static test much in the spirit of Section 5.2.

6 Correlation Risk

So far, we have modeled the financial system's aggregate risk as changes in the returns of individual banks' assets. In this section, we show the same economic forces apply if aggregate risk stems from the correlation in the returns of banks' portfolios. That is, we consider a cross-section of banks holding heterogeneous portfolios of risky assets, similar to Section 5, with known marginal distributions of returns but uncertain correlation. Each bank may know the distribution of its portfolio's returns but is uninformed about other banks' exact positions and hence does not know the correlation between these returns. Therefore, banks do not know how likely it is that other banks will need to sell their assets at the same they have to do it. Benoit, Colliard, Hurlin, and Pérignon (2016) provide a literature review documenting that systemic risk often stems from the correlated exposures between bank portfolios.

Modeling correlation risk as the source of aggregate uncertainty also provides a micro-foundation for the regulator's superior ability to forecast systemic risk. When the economic uncertainty stems from the correlation of individual banks' portfolios, the regulator has a natural advantage in assessing the quantity of aggregate risk over any outside observer or single bank by being able to monitor the cross-section of the banks' portfolios and evaluate their similarity.[41]

Moreover, correlation risk introduces a new general equilibrium feedback loop between capital requirements and asset prices in presence of fire sales. When faced with a severe negative shock, many banks have to sell assets at fire sale prices to cover their liabilities making it harder for the banks to remain solvent. Strict capital requirements reduce banks' exposure to such shocks and support higher asset prices by decreasing the likelihood and severity of future fire sales.

6.1 Embedding Correlation Risk into the Model

We extend our baseline model by an additional period $t = 3$ to incorporate the role of bank liquidity in asset prices at $t = 2$. There is a continuum of banks $i \in [0, 1]$. Each bank has b units of the safe bond, $d > b$ units of debt both maturing in period $t = 2$, and a units of the tradable asset that pays 1 unit of the numeraire good in period $t = 3$.[42] The riskiness of bank i stems from the non-tradable, i.e., perfectly illiquid, risky cash flow Y_i that realizes in period $t = 2$. Depending on the realization of $\theta \in \{0, 1\}$, the fraction μ_θ of banks carry idiosyncratic risk, and fraction $1 - \mu_\theta$ carry correlated risk, captured by

$$Y_i = \begin{cases} \xi_i & \text{with probability} \quad \mu_\theta, \\ Y & \text{with probability} \quad 1 - \mu_\theta, \end{cases}$$

where random variables $\{Y, (\xi_i)_{i\in[0,1]}\}$ are independent and uniformly distributed over $[0, 1]$ and $\mu_1 > \mu_0$, just like in Section 5. The aggregate risk of the financial system is captured by the fraction of the banks $1 - \mu_\theta$ that hold correlated risk. We keep the banks ex-ante identical and assume that it is unknown whether a particular bank holds systematic risk until period $t = 2$. However, the level of systemic risk θ is realized at $t = 1$, putting the regulator in a unique position to uncover θ from the cross-sectional similarity of the banks' portfolios *before* cash flows Y_i are

[41] Duffie (2010) advocates for collection of exposures of the largest banks to the stress scenarios to evaluate how correlated their losses are in these scenarios to establish a forward-looking metric for systemic risk.

[42] It is possible to incorporate tradable idiosyncratic bank risk, in the spirit of Section 5, by replacing the safe cash flow in period $t = 3$ with a risky cash flow.

realized at $t = 2$. By the Exact Law of Large Numbers (Duffie and Sun (2012)), The regulator can forecast the correlation between the banks' portfolios by computing the average similarity of the banks' period $t = 2$ risky cash flows for any *hypothetical* outcome $\omega \in \Omega$, which can be identified with the regulator conducting a scenario test on the cross-section of the banks' portfolios:[43]

$$\underbrace{\int_0^1 \int_0^1 \mathbb{1}\left\{Y_i(\omega) = Y_j(\omega)\right\} \, di \, dj}_{\substack{\textit{cross-sectional portfolio} \\ \textit{similarity for a hypothetical } \omega \in \Omega}} \overset{P-a.s.}{=} \mathrm{P}_\theta(Y_i = Y_j)^2 = (1 - \mu_\theta)^2 = \underbrace{\mathrm{corr}(Y_i, Y_j)}_{\substack{\textit{realized correlation} \\ \textit{between cash flows}}}. \tag{11}$$

The conceptual insight of (11) is that the regulator is well-positioned to forecast θ by exploiting her privileged private access to the cross-section of banks' portfolios.[44]

Banks improve their risk-weighted capital ratios by selling some of their 3-period bond at $t = 1$ to short-term risk-neutral investors who consume in period $t = 2$. If these investors own any long-term assets at $t = 2$, they sell them to the banks in exchange for the numeraire good. We assume the total quantity of the long-term tradable bond in the economy is $n \geq a$. For tractability, we restrict attention to stress tests that maximize social welfare subject to keeping the banks default-free.

6.2 Fire Sales and Feedback Effect

Bank balance sheet risk stems from individual cash flow shocks Y_i but is amplified by the possibility of fire sales occurring when a large number of banks become liquidity constrained at the same time. Tougher capital requirements reduce the severity of such sales and, as a result, increase ex-ante asset prices. To illustrate the interaction between correlation risk and capital requirements, we first derive asset prices under a fully opaque static stress test, i.e., one in which $S = \emptyset$.

Denote by A to be the maximum quantity of the long-term bond that the regulator allows the bank to retain[45] and by $p_1(A; \pi)$ to be the endogenously determined equilibrium price of the long-term tradable asset at $t = 1$ given retention quantity A and belief π. In period $t = 2$, banks suffering

[43]Under the stylized structure of Y_i, the regulator can acquire all of the information about θ just by running a single scenario. While we assume the regulator can manage this acquired information with commitment, it's possible to modify the underlying probability space structure to map partially informative signals with stress test scenarios of different magnitudes of adversity, similar to Section 3.2.

[44]The banks, on the other hand, are likely to be in a better position to learn θ from the time-series of their own portfolio. We assume that this is already incorporated in the common prior π_0.

[45]Similar to Section 2 there is a one-to-one mapping between retention A and the bank's risk-weighted capital ratio R.

negative shocks (low Y_i) sell their long-term bond to the market. The banks who receive positive shocks (high Y_i) pay off their debt d and buy the long-term bond from investors and other banks. Since investors consume only in period $t = 2$, the price $p_2(A, \theta, Y)$ of the long-term bond in period 2 in state θ given systematic shock Y is pinned down by the minimum between its face value of 1 and the cash in the market price price $p_2^C(A, \theta, Y)$ given by

$$n \cdot p_2^C(A, \theta, Y) \leq b + (a - A) \cdot p_1(A; \pi) + A \cdot p_2(A, \theta, Y) - d + \underbrace{\int_0^1 Y_i \, di}_{\mu_\theta/2 + (1-\mu_\theta) \cdot Y}.$$

Assumption 1. *The quantity of systematic risk* $1 - \mu_\theta$ *is sufficiently low if* $\theta = 1$ *so that* $p_2^C(a, 1, Y) \geq 1$*, and sufficiently high if* $\theta = 0$*, so that* $p_2^C(0, 0, 0) < 1$*, captured by* $\mu_1 > 2n > \mu_0$*.*

Assumption 1 guarantees that when $\theta = 1$ the long-term bond is trading at face of 1 in state $\theta = 1$ for any shock Y. If, however, $\theta = 0$, then the price of the long-term bond falls below fundamentals for a sufficiently severe aggregate shock, i.e., a sufficiently low realization of Y. The price of the risky asset at $t = 1$ is given by the investors' resale expectation at $t = 2$ as

$$p_1(A; \pi) = \delta \cdot \Big(\pi + (1 - \pi) \cdot \mathrm{E} \Big[\min \Big\{ p_2^C(A, 0, Y), \ 1 \Big\} \Big] \Big). \tag{12}$$

The next lemma shows the feedback effect between the bank's exposure A, and the price $p_1(A, \pi)$.

Lemma 5 (Feedback Effect)**.** *The period* $t = 1$ *price of the risky asset* $p_1(A; \pi)$ *is decreasing in the bank's exposure to risky assets* A *if and only if the sale discount* δ *at* $t = 1$ *is sufficiently high relative to the probability of fire sales* $\mathrm{P}\big(p_2^C(A; 0, Y) < 1\big)$ *in period* $t = 2$*.*

The dependence of $p_1(A; \pi)$ on A is obtained from analyzing the fixed point condition (12). Selling some of the risky assets at $t = 1$, i.e., reducing A, increases the banks' liquidity position at $t = 2$ if the discount suffered when obtaining the certainty equivalent is not too unfavorable. Hence for a fixed level of aggregate shock Y, the banks suffering such shock need to raise less from the market. Moreover, the banks having good idiosyncratic realizations of Y_i have more cash to buy assets. This force makes the fire sale less severe and supports on average higher asset prices $p_2(A, 0, Y)$. As a result, the time $t = 1$ price $p_1(A; \pi)$ goes up as well. An increase in the period $t = 1$ price increases the initial sale proceeds $(a - A)p_1(A;\pi)$ and further improves the liquidity position of the bank which. This, in turn, results in less severe fire sales and so on. The feedback effect from tighter capital requirements (lower A) to higher asset prices is the critical difference between the current section and Section 5 of this paper.

6.3 Optimal Stress Test

The optimal static stress test signal S balances the benefits of not overcapitalizing the banks if $\theta = 1$ and maintaining sufficient opacity to let the bank recapitalize if $\theta = 0$. Precautionary recapitalization is strictly welfare improving whenever there is a possibility of fire sales in period $t = 2$, leading the subsequent stress test fully informative.

Proposition 5. *A default-free stress test exists if and only if*

$$\pi_0 \geq \pi^c_{DF} \stackrel{def}{=} \max \left[1 - 2 \cdot \frac{1 - \mu_0}{\delta(1 - \mu_0/2n)^2} \cdot \frac{b + a\delta - d}{a \cdot n}, \quad 0 \right].$$

The optimal ***static test*** *is adverse pass-fail with* $\mathrm{P}(\theta = 1 \,|\, S = \mathit{fail}) = \pi^c_{DF}$ *with the associated capital requirement* $R(\mathit{fail}) = 1$*. Moreover, the optimal* ***precautionary recapitalization*** *is such that the subsequent stress test is fully informative about* θ*.*

The optimal stress test maintains the stability of the financial system by managing the expected level of portfolio correlation among the banks. The risk-weighted capital ratio of the failing banks maps to a portfolio with no risky bonds, as their $t = 2$ price adds exposure to the systematic risk already held by the banks on their balance sheet. Two opposing forces pin down the optimal adversity of the stress test scenario. On the one hand, in order to maintain the solvency of the banks in state $\theta = 0$, the scenario needs to be more adverse, just like discussed in Section 3. On the other hand, imposing strict capital requirements on the banks failing the stress test leads them to raise capital, thus reducing the likelihood and severity of the fire sale relative to the counter-factual fire sales if the banks were to keep their original portfolios b and a. This macro-prudential nature of the test, which accounts for the pricing implications of stricter capital requirements, reduces the stress test's optimal adversity and leads to a more transparent outcome relative to Section 5.

In Section 4 we have shown that precautionary recapitalization is welfare improving as it improves ex-ante risk-sharing without the costly over-capitalization in state $\theta = 1$. In the presence of fire sales, precautionary recapitalization (and ex-ante risk-sharing) additionally benefit the price of the risky bond. Stronger banks' portfolios endogenously reduce the riskiness of the $\theta = 0$ state (by decreasing the likelihood and severity of fire sales) and, as a result, the necessary amount of capital the banks need to raise at $t = 1$ after the stress test. Moreover, a higher expected price p_2 increases the price of risky bonds in period 1, making it easier for the banks to raise capital. As a result, precautionary recapitalization is strictly welfare improving for all beliefs as long as there is a possibility of fire sales in period $t = 2$. The general optimality of precautionary recapitalization whenever there is possibility of distress resonates with Proposition 2 given the absence of transaction costs in this setting.

7 Conclusion

A stress test is a forward-looking tool used to identify emerging risks and prepare the financial system for their eventuality. Information policy and capital requirements work together in ensuring that banks are safe going forward at the lowest cost to the banking system and society. We show that the optimal test relies on an adverse scenario that weak but also some strong banks fail. When there is a lot of risk the regulator imposes strict capital requirements. Precautionary recapitalization reduces the stigma of failing the stress test and allows the regulator to conduct a more informative stress test. As a result, it improves risk-sharing and ex-post asset allocation at the same time. We show that under optimal precautionary recapitalization the subsequent stress test is fully informative and the need for commitment to opacity is eliminated. Furthermore, full precautionary recapitalization is optimal in a broad class of dynamic interventions we term sequential stress tests. Precautionary recapitalization also improves risk-sharing and reduces fire sales when banks are heterogeneous in their risk exposures. These results provide novel insights into the beneficial interaction between TARP in 2008 and SCAP in 2009, as well as provide policy guidance going forward.

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A Online Appendix

Contents

A.1 Proof of Lemma 1

Special case of Proposition 1 if $a = 0$ and $w = b$. The optimality of $\pi^* = 0$ for $a = 0$ can be seen, in the proof of Proposition 1, from (A.10), and the argument that immediately follows, always being satisfied for if $a = 0$.

A.2 Proof of Lemma 2

The regulator's welfare objective is given by (1) and given by

$$\textit{Social Welfare} = b + \hat{a} \cdot X + (a - \hat{a}) \cdot \delta X - c\big(\max\big[d - \hat{b} - \hat{a} \cdot X,\ 0\big]\big).$$

where the price of the risky asset is $p \overset{def}{=} \delta \mathrm{E}[X] = \delta \frac{1+\pi}{2}$ and the budget constraint is capture by the bu $\hat{b} \overset{def}{=} b + (a - \hat{a}) \cdot p$. Define

$$\hat{u}(x) \overset{def}{=} \begin{cases} x & \text{if} \quad x \geq 0, \\ x - c(x) & \text{if} \quad x < 0. \end{cases}$$

The expected welfare can then be written as

$$\hat{U}(w) = \max_{\hat{a}} \mathrm{E}\left[\hat{u}\Big(w + \hat{a} \cdot (X - p)\Big)\right], \tag{A.1}$$

where $w \stackrel{def}{=} b + a \cdot p$. The first order condition pinning down optimal asset holdings $A(w)$ in (A.1) is

$$\mathrm{E}\left[\hat{u}'\Big(w + A(w)\cdot(X-p)\Big)\cdot(X-p)\right] = 0.$$

Differentiating this expression with respect to w obtain

$$\mathrm{E}\left[\hat{u}''\Big(w + A(w)\cdot(X-p)\Big)\cdot\Big((X-p) + A'(w)\cdot(X-p)^2\Big)\right] = 0,$$
$$-\frac{\mathrm{E}\left[\hat{u}''(w + A(w)\cdot(X-p))\cdot(X-p)\right]}{\mathrm{E}\left[\hat{u}''(w + A(w)\cdot(X-p))\cdot(X-p)^2\right]} = A'(w).$$

Applying the Envelope theorem with respect to w in (A.1) obtain

$$\hat{U}'(w) = \mathrm{E}\left[\hat{u}'\Big(w + A(w)\cdot(X-p)\Big)\right]. \tag{A.2}$$

Differentiating (A.2) with respect to w obtain

$$\begin{aligned}\hat{U}''(w) &= \mathrm{E}\left[\hat{u}''\Big(w + A(w)\cdot(X-p)\Big)\cdot\Big(1 + A'(w)\cdot(X-p)\Big)\right]\\ &= \mathrm{E}\left[\hat{u}''\Big(w + A(w)\cdot(X-p)\Big)\cdot\left(1 - \frac{\mathrm{E}\left[\hat{u}''(w + A(w)\cdot(X-p))\cdot(X-p)\right]}{\mathrm{E}\left[\hat{u}''(w + A(w)\cdot(X-p))\cdot(X-p)^2\right]}\cdot(X-p)\right)\right]\end{aligned}$$

Define

$$\begin{cases}\xi \stackrel{def}{=} X - p,\\ \eta \stackrel{def}{=} \hat{u}''\Big(w + A(w)\cdot(X-p)\Big) \le 0.\end{cases} \tag{A.3}$$

Then

$$\begin{aligned}\hat{U}''(w) &= \mathrm{E}\left[\eta\cdot\left(1 - \frac{\mathrm{E}\left[\eta\cdot\xi\right]}{\mathrm{E}\left[\eta\cdot\xi^2\right]}\cdot\xi\right)\right] = \frac{\mathrm{E}\left[\eta\right]\cdot\mathrm{E}\left[\eta\cdot\xi^2\right] - \left(\mathrm{E}\left[\eta\cdot\xi\right]\right)^2}{\mathrm{E}\left[\eta\cdot\xi^2\right]}\\ &\stackrel{(i)}{=} \frac{\mathrm{E}\left[\eta\right]\cdot\mathrm{E}\left[\eta\cdot\xi^2\right] - \left(\mathrm{E}\left[|\eta|\cdot|\xi|\right]\right)^2}{\mathrm{E}\left[\eta\cdot\xi^2\right]} = \frac{\mathrm{E}\left[\eta\right]\cdot\mathrm{E}\left[\eta\cdot\xi^2\right] - \left(\mathrm{E}\left[\sqrt{|\eta|\cdot\xi^2}\cdot\sqrt{|\eta|}\right]\right)^2}{\mathrm{E}\left[\eta\cdot\xi^2\right]}\\ &= \frac{\mathrm{E}\left[\eta\right]\cdot\mathrm{E}\left[\eta\cdot\xi^2\right] - \mathrm{E}\left[|\eta|\cdot|\xi|^2\right]\cdot\mathrm{E}\left[|\eta|\right] + \mathrm{E}\left[|\eta|\cdot|\xi|^2\right]\cdot\mathrm{E}\left[|\eta|\right] - \left(\mathrm{E}\left[\sqrt{|\eta|\cdot\xi^2}\cdot\sqrt{|\eta|}\right]\right)^2}{\mathrm{E}\left[\eta\cdot\xi^2\right]}\\ &= \frac{\mathrm{E}\left[|\eta|\cdot|\xi|^2\right]\cdot\mathrm{E}\left[|\eta|\right] - \left(\mathrm{E}\left[\sqrt{|\eta|\cdot\xi^2}\cdot\sqrt{|\eta|}\right]\right)^2}{\mathrm{E}\left[\eta\cdot\xi^2\right]} \stackrel{(ii)}{\le} 0,\end{aligned}$$

where equality (i) holds because $\eta \cdot \xi^2 \leq 0$ and does not change sign, and (ii) holds by the Cauchy-Schwarz inequality. This implies that the value function $\hat{U}(w)$ is concave in wealth w.

A.3 Proof of Proposition 1

It is convenient to define $m \overset{def}{=} \mathrm{E}_0[X] = 1/2$ and define

$$H(x) \overset{def}{=} \begin{cases} 0 & \text{if} \quad x > 0, \\ -c(-x) & \text{if} \quad x < 0. \end{cases}$$

Denote the expected social welfare, net of the bank's ex-ante balance sheet value $b + a \cdot \delta \mathrm{E}[X]$, as the utility from retaining $\hat{a}$ units of the risky asset is given by

$$v(\pi, \hat{a}) \overset{def}{=} \hat{a}(1-\delta)\Big(\pi + (1-\pi)m\Big) + (1-\pi) \cdot \mathrm{E}_0\left[H\Big(b - d + (a-\hat{a})\delta\big(\pi + (1-\pi)m\big) + \hat{a}X\Big)\right].$$

Throughout the proofs we refer to $v(\pi, \hat{a})$ as the regulator's "utility" as she internalizes the expected social welfare. The derivative of $v(\pi, \hat{a})$ with respect to $\hat{a}$ is

$$\begin{aligned} \frac{\partial v}{\partial \hat{a}}(\pi, \hat{a}) &= (1-\delta)(\pi + (1-\pi)m) \\ &\quad + (1-\pi)\mathrm{E}_0\left[H'\Big(b - d + (a-\hat{a})\delta\big(\pi + (1-\pi)m\big) + \hat{a}X\Big)\Big(X - \delta(\pi + (1-\pi)m)\Big)\right] \end{aligned} \tag{A.4}$$

Define $v(\pi) \overset{def}{=} \max_{\hat{a}} v(\pi, \hat{a})$ to be the value function under the optimal portfolio and $A(\pi) = \arg\max_{\hat{a}} v(\pi, \hat{a})$ to be the optimal retention of the risky asset.

Lemma A.1. *The optimal test probabilistically reveals $\theta = 1$ for any $\pi \in [\pi_{DF}, 1]$ if $\forall \pi \in [\pi_{DF}, 1]$*

$$v'(\pi-) \leq \frac{v(1) - v(\pi)}{1-\pi}. \tag{A.5}$$

Proof. Consider an ε disclosure of $\theta = 1$. The gain from such a disclosure is

$$\varepsilon \cdot v(1) + (1-\varepsilon) \cdot v\left(\frac{\pi - \varepsilon}{1-\varepsilon}\right) - v(\pi) = \varepsilon\left(v(1) - v(\pi)\right) + (1-\varepsilon)\left(v\left(\frac{\pi-\varepsilon}{1-\varepsilon}\right) - v(\pi)\right)$$

$$= \varepsilon \cdot (v(1) - v(\pi)) + (1-\varepsilon) \cdot v'(\pi-) \cdot \left(\frac{\pi - \varepsilon}{1-\varepsilon} - \pi\right) + O(\varepsilon^2)$$
$$= \varepsilon \cdot (v(1) - v(\pi)) + (1-\varepsilon) \cdot v'(\pi-) \cdot \left(\frac{\pi\varepsilon - \varepsilon}{1-\varepsilon}\right) + O(\varepsilon^2)$$
$$= \varepsilon \cdot \big(v(1) - v(\pi) - v'(\pi-) \cdot (1-\pi)\big) + O(\varepsilon^2).$$

Thus, disclosure of $\theta = 1$ is valuable for the regulator whenever $v'(\pi-) \leq \frac{v(1)-v(\pi)}{1-\pi}$. □

Optimality of information revelation over $[\pi^*, 1]$ for $\pi^* \leq \pi_{DF}$

Suppose the budget constraint is not binding, i.e., $A(\pi)$ satisfies the binding first-order optimality condition $\frac{\partial v}{\partial \hat{a}}(\pi, A(\pi)) = 0$. Applying Envelope theorem, differentiate $v(\pi)$ with respect to π to obtain[46]

$$v'(\pi-) = A(\pi)(1-\delta)(1-m) - \mathrm{E}_0\left[H\left(b - d + (a - A(\pi))\delta\big(\pi + (1-\pi)m\big) + A(\pi)X\right)\right]$$
$$+ (1-\pi)(a - A(\pi))\delta(1-m)\mathrm{E}_0\left[H'_-\left(b - d + (a - A(\pi))\delta\big(\pi + (1-\pi)m\big) + A(\pi)X\right)\right] \quad \text{(A.6)}$$

Rewrite (A.5) using (A.6) as

$$(1-\pi)v'(\pi) + v(\pi) - v(1)$$

$$=A(\pi)(1-\delta)(1-m)(1-\pi) - (1-\pi) \cdot \mathrm{E}_0\left[H\left(b - d + (a - A(\pi))\delta\big(\pi + (1-\pi)m\big) + A(\pi)X\right)\right]$$
$$+(1-\pi)^2(a - A(\pi))\delta(1-m) \cdot \mathrm{E}_0\left[H'\left(b - d + (a - A(\pi))\delta\big(\pi + (1-\pi)m\big) + A(\pi)X\right)\right]$$
$$+A(\pi)(1-\delta)(\pi + (1-\pi)m) - \left(a + \frac{b}{\delta}\right)(1-\delta)$$
$$+(1-\pi) \cdot \mathrm{E}_0\left[H\left(b - d + (a - A(\pi))\delta\big(\pi + (1-\pi)m\big) + A(\pi)X\right)\right]$$

$$= -(a - A(\pi))(1-\delta) - \frac{b}{\delta}(1-\delta)$$
$$+(a - A(\pi))(1-\pi)^2\delta(1-m) \cdot \mathrm{E}_0\left[H'_-\left(b - d + (a - A(\pi))\delta\big(\pi + (1-\pi)m\big) + A(\pi)X\right)\right]$$

[46] We use notation $H'_-(f(\pi))$ to denote $\frac{1}{f'(\pi)} \cdot \frac{\partial}{\partial \pi-} H(f(\pi))$.

Condition $(1-\pi)v'(\pi-)+v(\pi)-v(1)\leq 0$ can then be expressed as

$$(a-A(\pi))(1-\pi)^2(1-m)\mathrm{E}_0\left[H'_-\left(b-d+(a-A(\pi))\delta\left(\pi+(1-\pi)m\right)+A(\pi)X\right)\right]\leq\frac{1-\delta}{\delta}\left(a-A(\pi)+\frac{b}{\delta}\right).$$

Suppose $A(\pi)>a$ then $H'_-(\ldots)=H'(\ldots+0)>0$ and we need to show that

$$(a-A(\pi))(1-\pi)^2(1-m)\mathrm{E}_0\left[H'(\ldots+0)\right]\leq\frac{1-\delta}{\delta}\left(a-A(\pi)+\frac{b}{\delta}\right). \tag{A.7}$$

If $a+\frac{b}{\delta}=A(1)\geq A(\pi)>a$ then the inequality clearly holds, as the l.h.s. is negative and the r.h.s. is positive. Next, consider the case $A(\pi)>A(1)>a$ which turns the inequality into

$$\delta\frac{A(\pi)-a}{A(\pi)-A(1)}(1-\pi)^2(1-m)\mathrm{E}_0\left[H'(\ldots+0)\right]\geq 1-\delta. \tag{A.8}$$

Given that the first-order optimality condition $\frac{\partial v}{\partial \hat{a}}(\pi,\hat{a})|_{\hat{a}=A(\pi)+}\leq 0$ is satisfied at π, rewrite (A.4) as

$$1-\delta\leq-\frac{1-\pi}{\pi+(1-\pi)m}\mathrm{E}_0\left[H'\left(b-d+(a-A(\pi))\delta\left(\pi+(1-\pi)m\right)+A(\pi)X+0\right)\left(X-\delta(\pi+(1-\pi)m)\right)\right].$$

Hence it is sufficient to show that

$$\delta\frac{A(\pi)-a}{A(\pi)-A(1)}(1-z)\mathrm{E}_0\left[H'(\ldots+0)\right]\geq-\frac{1}{z}\mathrm{E}_0\left[H'\left(\ldots+0\right)\left(X-\delta z\right)\right]$$

with $z=\pi+(1-\pi)m$, or equvalently

$$\mathrm{E}_0\left[H'\left(\ldots+0\right)\left(X-\delta z+\delta z(1-z)\frac{A(\pi)-a}{A(\pi)-A(1)}\right)\right]\geq 0. \tag{A.9}$$

The quantity of the asset held by the bank $A(\pi)$ is limited above by the liquidity constraint, i.e.,

$A(\pi) \le a + \frac{b}{\delta z}$ and as a result $\frac{A(\pi)-a}{A(\pi)-A(1)} \ge \frac{1}{1-z}$. Hence

$$X - \delta z + \delta z(1-z)\frac{A(\pi) - a}{A(\pi) - A(1)} \ge X - \delta z + \delta z(1-z)\frac{1}{1-z} = X$$

which implies

$$\mathrm{E}_0\left[H'\Big(\ldots + 0\Big)\left(X - \delta z + \delta z(1-z)\frac{A(\pi) - a}{A(\pi) - A(1)}\right)\right] \ge \mathrm{E}_0\left[H'\Big(\ldots + 0\Big)\cdot X\right] > 0.$$

Next, suppose $A(\pi) < a$ then $H'_{-}(\ldots) = H'(\ldots - 0) > 0$ and we need to show that

$$\frac{a - A(\pi)}{A(1) - A(\pi)}\delta(1-\pi)^2(1-m)\mathrm{E}_0\left[H'(\ldots - 0)\right] \le 1 - \delta.$$

Given that the first-order optimality condition $\frac{\partial v}{\partial \hat{a}}(\pi, \hat{a})|_{\hat{a}=A(\pi)-} \ge 0$ is satisfied at π, rewrite (A.4) as

$$1 - \delta \ge -\frac{1-\pi}{z}\mathrm{E}_0\left[H'\Big(\ldots - 0\Big)\Big(X - \delta z\Big)\right].$$

Hence it is sufficient to show that

$$\delta\frac{a - A(\pi)}{A(1) - A(\pi)}(1-z)\mathrm{E}_0\left[H'(\ldots - 0)\right] \le -\frac{1}{z}\mathrm{E}_0\left[H'\Big(\ldots - 0\Big)\Big(X - \delta z\Big)\right]$$

or, equivalently that

$$\mathrm{E}_0\left[H'\Big(\ldots - 0\Big)\left(X - \delta z^2 - \frac{b \cdot z(1-z)}{A(1) - A(\pi)}\right)\right] \le 0. \tag{A.10}$$

Define $\bar{x}$ as the break-even cash flow realization at which the bank is solvent at $t = 2$, given by

$$0 = b - d + (a - A(\pi)) \cdot \delta z + A(\pi) \cdot \bar{x},$$
$$\bar{x} \stackrel{def}{=} \frac{d - b + (A(\pi) - a) \cdot \delta z}{A(\pi)} = \delta z + \frac{d - b - a \cdot \delta z}{A(\pi)}.$$

Because X is uniform on $[0,1]$ it follows that

$$\mathrm{E}\left[X|X \leq \bar{x}\right] = \frac{\bar{x}}{2} = \frac{d-b+(A(\pi)-a)\delta z}{2A(\pi)}.$$

A sufficient condition for convexity can then be written as

$$\begin{aligned}
\frac{d-b+(A(\pi)-a)\delta z}{2A(\pi)} - \delta z^2 - \frac{b(z-z^2)}{a-A(\pi)+b/\delta} &\leq 0,\\
\frac{d-b+(A(\pi)-a)\delta z}{2A(\pi)} - \frac{(a-A(\pi))\delta z^2 + bz}{a-A(\pi)+b/\delta} &\leq 0,\\
\frac{d-b+(A(\pi)-a)\delta z}{2A(\pi)} + \frac{(A(\pi)-a)\delta z^2 - bz}{a-A(\pi)+b/\delta} &\leq 0,\\
\frac{d-b-a\delta z}{2A(\pi)} + \frac{\delta z}{2} + \frac{(A(\pi)-a)\delta z^2 - bz}{a-A(\pi)+b/\delta} &\leq 0,\\
\frac{d-b-a\delta z}{2A(\pi)} + \frac{(a-A(\pi))\left(\frac{\delta z}{2}-\delta z^2\right) - lz/2}{a-A(\pi)+b/\delta} &\leq 0,\\
\frac{d-b-a\delta z}{2A(\pi)} + \frac{(a-A(\pi))\delta z\left(\frac{1}{2}-z\right) - bz/2}{a-A(\pi)+b/\delta} &\leq 0. \qquad \text{(A.11)}
\end{aligned}$$

The first term in (A.11) is negative whenever $d-b-a\delta z \leq 0$, i.e., whenever $\pi \geq \pi_{DF}$. The second term is always negative if $z \geq \frac{1}{2}$ which is always the case since $z = \pi + (1-\pi)m = \frac{1+\pi}{2}$. Hence we can bound the l.h.s. in (A.10) as follows

$$\begin{aligned}
&\mathrm{E}_0\left[H'_-\left(b-d+(a-A(\pi))\delta z + A(\pi)X\right)\left(X-\delta z^2-\frac{b\cdot z(1-z)}{a-A(\pi)+b/\delta}\right)\right]\\
&= \mathrm{P}(X\leq\bar{x})\cdot \mathrm{E}_0\left[H'_-\left(b-d+(a-A(\pi))\delta z + A(\pi)X\right)\left(X-\delta z^2-\frac{b\cdot z(1-z)}{a-A(\pi)+b/\delta}\right) \mid X \leq \bar{x}\right]\\
&\leq \mathrm{P}(X\leq\bar{x})\cdot \mathrm{E}_0\left[H'_-\left(b-d+(a-A(\pi))\delta z + A(\pi)\bar{x}\right)\left(X-\delta z^2-\frac{b\cdot z(1-z)}{a-A(\pi)+b/\delta}\right) \mid X \leq \bar{x}\right]\\
&\leq \mathrm{P}(X\leq\bar{x})\cdot H'_-\left(b-d+(a-A(\pi))\delta z + A(\pi)\bar{x}\right)\cdot \mathrm{E}_0\left[X-\delta z^2-\frac{b\cdot z(1-z)}{a-A(\pi)+b/\delta} \mid X \leq \bar{x}\right]\\
&\leq 0
\end{aligned}$$

Hence, following Lemma A.1 that it is optimal for the regulator to disclose $\theta = 1$ as long as $\pi \geq \pi_{DF}$.[47]

[47]Note that we show that disclosure of $[\pi_{DF}, 1]$ is always optimal for the regulator. However the optimal point at which the regulator should no longer disclose information is $\pi^* \leq \pi_{DF}$ with a, possibly strict, inequality depending

Now, suppose that the bank's budget constraint is binding, i.e., $A(\pi) = a + \frac{b}{\delta(\pi+(1-\pi)m)}$ and $\frac{\partial v}{\partial \hat{a}}(\pi, \hat{a})|_{\hat{a}=A(\pi)} > 0$. In this case, the regulator's value is given by

$$v(\pi) = b \cdot (1-\delta) + a\frac{1+\pi}{2} \cdot (1-\delta) + (1-\pi) \cdot \mathrm{E}\left[H\left(\left(a + \frac{b}{\delta(1+\pi)/2}\right)X - d\right)\right]$$

The derivative with respect to π is given by

$$\begin{aligned} v'(\pi) = a\frac{1-\delta}{2} &- \mathrm{E}\left[H\left(\left(a + \frac{b}{\delta(1+\pi)/2}\right)X - d\right)\right] \\ &- (1-\pi) \cdot \mathrm{E}\left[H'\left(\left(a + \frac{b}{\delta(1+\pi)/2}\right)X - d\right) \cdot X \cdot \frac{2b}{\delta(1+\pi)^2}\right]. \end{aligned}$$

The desired convexity condition then takes form

$$\begin{aligned} &(1-\pi)v'(\pi) + v(\pi) - v(1) \\ =&a\frac{1-\delta}{2}(1-\pi) - (1-\pi)^2 \cdot \mathrm{E}\left[H'\left(\left(a + \frac{b}{\delta(1+\pi)/2}\right)X - d\right) \cdot X \cdot \frac{2b}{\delta(1+\pi)^2}\right] \\ +&b(1-\delta) + a \cdot \frac{1+\pi}{2}(1-\delta) - b(1-\delta) - a \cdot \frac{1-\delta}{2} \\ =&-(1-\pi)^2 \cdot \mathrm{E}\left[H'\left(\left(a + \frac{b}{\delta(1+\pi)/2}\right)X - d\right) \cdot X \cdot \frac{2b}{\delta(1+\pi)^2}\right] < 0, \end{aligned}$$

implying that, following Lemma A.1 that it is optimal for the regulator to disclose $\theta = 1$ whenever the budget constraint is binding.

Sufficient condition for optimality of information pooling at $\pi^* > 0$

A sufficient condition is that at $\pi = 0$ it is optimal not to disclose all information

$$v'(0) \geq v(1) - v(0). \tag{A.12}$$

on the severity of the distress cost.

Suppose the first-order optimality condition $\frac{\partial}{\partial \hat{a}} u(0, \hat{a})|_{\hat{a}=A(0)} = 0$ is binding. It is possible to express (A.12) by rewriting (A.10) as

$$\begin{cases} \mathrm{E}_0 \left[H' \left(b - d + a \cdot \frac{\delta}{2} + A(0) \cdot \left(X - \frac{\delta}{2} \right) \right) \cdot \left(X - \frac{\delta}{2} \right) \right] = -\frac{\delta}{2}, \\ \mathrm{E}_0 \left[H' \left(b - d + a \cdot \frac{\delta}{2} + A(0) \cdot \left(X - \frac{\delta}{2} \right) \right) \cdot \left(X - \frac{\delta}{4} - \frac{b/4}{a - A(0) + b/\delta} \right) \right] \overset{(i)}{\geq} 0. \end{cases} \tag{A.13}$$

Inequality (i) in (A.13) holds if

$$\begin{aligned} -\frac{\delta}{4} - \frac{b/4}{a - A(0) + b/\delta} &\geq 0 \\ \delta + \frac{b}{a - A(0) + b/\delta} &\leq 0 \\ \frac{a - A(0) + 2b/\delta}{a - A(0) + b/\delta} &\leq 0. \end{aligned} \tag{A.14}$$

Note that $A(0) \leq a + 2b/\delta$, implying that (A.14) holds whenever $a - A(0) + b/\delta \leq 0$.

Consider, now, the remaining case of $a - A(0) + b/\delta \geq 0$. Subtracting the first equation from the second inequality in (A.13), obtain that pooling at $\pi^* > 0$ is optimal if

$$\begin{aligned} \mathrm{E}_0 \left[H' \left(b - d + a \cdot \frac{\delta}{2} + A(z) \cdot \left(X - \frac{\delta}{2} \right) \right) \cdot \left(\frac{\delta}{4} - \frac{b/4}{a - A(0) + b/\delta} \right) \right] &\geq \frac{\delta}{2} \\ \mathrm{E}_0 \left[H' \left(b - d + a \cdot \frac{\delta}{2} + A(0) \cdot \left(X - \frac{\delta}{2} \right) \right) \cdot \left(\frac{1}{2} - \frac{\frac{b}{2\delta}}{a - A(0) + b/\delta} \right) \right] &\geq 1 \\ \mathrm{E}_0 \left[H' \left(b - d + a \cdot \frac{\delta}{2} + A(0) \cdot \left(X - \frac{\delta}{2} \right) \right) \cdot \frac{a - A(0)}{a - A(0) + b/\delta} \right] &\geq 2 \\ \mathrm{E}_0 \left[H' \left(b - d + a \cdot \frac{\delta}{2} + A(0) \cdot \left(X - \frac{\delta}{2} \right) \right) \cdot \left(a - A(0) \right) \right] &\geq 2 \cdot \left(a - A(0) + \frac{b}{\delta} \right) \end{aligned}$$

The condition above is sufficient (albeit implicit) for an arbitrary default cost H and the corresponding pooling threshold π^*. In Lemma 3 we provide explicit conditions for the model's primitive parameters when $\pi^* = \pi_{DF}$.

A.4 Proof of Lemma 3

Suppose the marginal cost of distress $H'(0)$ is high relative to the bank's net equity $b + a\delta - d$

$$H'(0) \geq \bar{c} \stackrel{def}{=} \max\left\{\frac{1-\delta}{b+a\delta-d}\cdot\frac{a}{d-b}, \quad \frac{1-\delta}{3}\cdot\frac{a}{b+a\delta-d}\cdot\frac{a}{d-b}\cdot\left(\frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}\right)\right\}$$
$$= \frac{1-\delta}{b+a\delta-d}\cdot\frac{a}{d-b}\cdot\max\left\{1, \quad \frac{a}{3}\cdot\left(\frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}\right)\right\}, \tag{A.15}$$

which is obtained as a superposition of constraints (A.16) and (A.24). Then, the optimal static stress test induces no bank distress. The proof proceeds in two steps. First, we show that at $\pi = \pi_{DF}$, it is optimal to choose a riskless portfolio. Second, we show that the optimal concavification point is $\pi^* = \pi_{DF}$.

Lemma 3, part 1: optimal riskless portfolio at $\pi = \pi_{DF}$

Suppose $b + a \cdot \delta\frac{1+\pi}{2} = d$, which constitutes the definition of π_{DF}. A $\varepsilon > 0$ holding of the risky asset generates an expected benefit

$$\varepsilon(1-\delta)\frac{1+\pi}{2} + (1-\pi)\cdot \mathrm{E}\left[H\left(b + (a-\varepsilon)\cdot\delta\frac{1+\pi}{2} + \varepsilon\cdot X - d\right)\right]$$
$$= \varepsilon(1-\delta)\frac{1+\pi}{2} + (1-\pi)\cdot\int_0^{\delta\frac{1+\pi}{2}} H\left(\varepsilon\left(x - \delta\frac{1+\pi}{2}\right)\right)dx$$
$$= \varepsilon(1-\delta)\frac{1+\pi}{2} + (1-\pi)\cdot\int_0^{\frac{d-b}{a}} H'(0)\cdot\varepsilon\left(x - \frac{d-b}{a}\right)dx + \overline{o}(\varepsilon)$$
$$= \varepsilon\frac{1-\delta}{\delta}\cdot\frac{d-b}{a} - \varepsilon\cdot(1-\pi)\cdot H'(0)\cdot\frac{1}{2}\left(\frac{d-b}{a}\right)^2 + \overline{o}(\varepsilon).$$

This implies a marginal increase in the risky asset holding is unprofitable at $\pi = \pi_{DF}$ if

$$\frac{1-\delta}{\delta}\cdot\frac{d-b}{a} - (1-\pi_{DF})\cdot H'(0)\cdot\frac{1}{2}\left(\frac{d-b}{a}\right)^2 \leq 0$$
$$\frac{1-\delta}{\delta} \leq (1-\pi_{DF})\cdot H'(0)\cdot\frac{1}{2}\cdot\frac{d-b}{a}$$
$$\frac{1-\delta}{\delta^2} \leq \frac{H'(0)}{4}\cdot\left(1-\pi_{DF}^2\right)$$
$$\frac{4}{H'(0)}\cdot\frac{1-\delta}{\delta^2} \leq 1 - \pi_{DF}^2$$

$$\pi_{DF}^2 \leq 1 - \frac{4}{H'(0)} \cdot \frac{1-\delta}{\delta^2}$$
$$\pi_{DF} \leq \sqrt{1 - \frac{4}{H'(0)} \cdot \frac{1-\delta}{\delta^2}}.$$

Equivalently, the sufficient condition for optimality of 0 asset holdings at $\pi = \pi_{DF}$ is

$$H'(0) \geq \frac{4(1-\delta)}{(1-\pi_{DF}^2)\delta^2} = \frac{1-\delta}{b + a\delta - d} \cdot \frac{a}{d-b}. \tag{A.16}$$

Now, consider an arbitrary cost function $H(\cdot)$. Then, the payoff from retaining $\hat{a}$ units of the asset if $\pi = \pi_{DF}$ is given by

$$\hat{a}(1-\delta)\frac{1+\pi}{2} + (1-\pi)\int_0^{\frac{d-b}{a}} H\left(\hat{a}\left(x - \delta\frac{1+\pi_{DF}}{2}\right)\right) dx \overset{(i)}{\leq}$$
$$\hat{a}(1-\delta)\frac{1+\pi}{2} + (1-\pi)\int_0^{\frac{d-b}{a}} H'(0) \cdot \hat{a}(x-p)\, dx =$$
$$\hat{a}\left[(1-\delta)\frac{1+\pi}{2} - (1-\pi)H'(0) \cdot \frac{1}{2}\left(\delta\frac{1+\pi_{DF}}{2}\right)^2\right] \leq 0$$

for any $a \geq 0$, where (i) follows from the convexity of the cost function $H(\cdot)$.

Lemma 3, part 2: optimality of information pooling at $\pi^* = \pi_{DF}$

If the default-free test discloses information between $[\pi_{DF}, 1]$, then the slope of the regulator's value function at $\pi = \pi_{DF}$ is given by

$$\frac{v(1) - v(\pi_{DF})}{1 - \pi_{DF}} = \frac{\left(\frac{b}{\delta} + a\right)(1-\delta)}{1 - \pi_{DF}}.$$

The necessary and sufficient condition for $\pi^* = \pi_{DF}$ is that value function $v(\pi)$ experiences a downward kink at $\pi = \pi_{DF}$, i.e., $\forall \pi < \pi_{DF}$

$$\frac{v(\pi_{DF}) - v(\pi)}{\pi_{DF} - \pi} \geq \frac{v(1) - v(\pi_{DF})}{1 - \pi_{DF}}$$
$$\frac{v(\pi)}{\pi_{DF} - \pi} \leq -\frac{(b/\delta + a)(1-\delta)}{1 - \pi_{DF}}, \tag{A.17}$$

where (A.17) follows since $v(\pi_{DF}) = 0$. Define cost function $\hat{H}(x)$ as

$$\hat{H}(x) \overset{def}{=} \begin{cases} 0 & \text{if} \quad x \geq 0, \\ H'(0) \cdot x & \text{if} \quad x < 0. \end{cases}$$

By construction, $\hat{H}(x) \geq H(x)$. Then

$$\begin{aligned} v(\pi) &\overset{(i)}{=} \max_{\hat{a}} \left\{ \hat{a}(1-\delta)\frac{1+\pi}{2} + (1-\pi) \cdot \mathrm{E}\left[H\left(b - d + a \cdot \delta\frac{1+\pi}{2} + \hat{a} \cdot \left(X - \delta\frac{1+\pi}{2} \right) \right) \right] \right\} \\ &\overset{(ii)}{\leq} \max_{\hat{a}} \left\{ \hat{a}(1-\delta)\frac{1+\pi}{2} + (1-\pi) \cdot \mathrm{E}\left[\hat{H}\left(b - d + a \cdot \delta\frac{1+\pi}{2} + \hat{a} \cdot \left(X - \delta\frac{1+\pi}{2} \right) \right) \right] \right\} \overset{def}{=} \hat{v}(\pi). \end{aligned}$$

where (i) follows from the definition of $A(\pi)$, (ii) follows from the fact that $\hat{H}(x) \leq H(x)$ for any x, and we denoted by $\hat{v}(\pi)$ to be the regulator's value function corresponding to the linear cost function $\hat{H}(x)$. We proceed by establishing sufficient conditions under which

$$\hat{v}(\pi) \leq -\frac{\pi_{DF} - \pi}{1 - \pi_{DF}} \cdot \left(\frac{b}{\delta} + a \right)(1-\delta).$$

for $\pi < \pi_{DF}$. It follows from Lemma A.21 that

$$\bar{v}(\pi) = \left(d - b - a\delta\frac{1+\pi}{2} \right) \cdot \phi \left[(1-\delta)\frac{1+\pi}{2} - \frac{1-\pi}{2} H'(0) \left(\delta\frac{1+\pi}{2} + \frac{1}{\phi} \right)^2 \right],$$

where $\phi \overset{def}{=} \frac{\sqrt{H'(0) \cdot \frac{1-\pi}{1+\pi}}}{\sqrt{H'(0) \cdot \delta^2 \cdot \frac{1-\pi^2}{4} - (1-\delta)}}$. Then, it is sufficient to show that $\bar{v}(\pi) \leq -\frac{\pi_{DF}-\pi}{1-\pi_{DF}} \cdot \left(\frac{b}{\delta} + a\right)(1-\delta)$, which follows from a sequence of derivations

$$\begin{aligned} \left(d - b - a\delta\frac{1+\pi}{2} \right) \cdot \phi \left[(1-\delta)\frac{1+\pi}{2} - \frac{1-\pi}{2} H'(0) \left(\delta\frac{1+\pi}{2} + \frac{1}{\phi} \right)^2 \right] &\leq -\frac{\pi_{DF} - \pi}{1 - \pi_{DF}} \left(\frac{b}{\delta} + a \right)(1-\delta) \\ a\delta\frac{\pi_{DF} - \pi}{2} \cdot \phi \left[(1-\delta)\frac{1+\pi}{2} - \frac{1-\pi}{2} H'(0) \left(\delta\frac{1+\pi}{2} + \frac{1}{\phi} \right)^2 \right] &\leq -\frac{\pi_{DF} - \pi}{1 - \pi_{DF}} \left(\frac{b}{\delta} + a \right)(1-\delta) \\ \frac{a\delta}{2} \cdot \phi \left[\frac{1+\pi}{2} - \frac{1-\pi}{2} \frac{H'(0)}{1-\delta} \left(\delta\frac{1+\pi}{2} + \frac{1}{\phi} \right)^2 \right] &\leq -\frac{1}{1 - \pi_{DF}} \left(\frac{b}{\delta} + a \right) \\ \phi \left[1 + \pi - (1-\pi)\frac{H'(0)}{1-\delta} \left(\delta\frac{1+\pi}{2} + \frac{1}{\phi} \right)^2 \right] &\leq -\frac{4}{a\delta} \frac{1}{1 - \pi_{DF}} \left(\frac{b}{\delta} + a \right) \end{aligned}$$

$$1+\pi-(1-\pi)\frac{H'(0)}{1-\delta}\left(\left(\delta\frac{1+\pi}{2}\right)^2\cdot\phi+\delta\frac{1+\pi}{2}+\frac{1}{\phi}\right)\leq-\frac{4}{a\delta}\frac{1}{1-\pi_{DF}}\left(\frac{b}{\delta}+a\right)$$

$$1+\pi-H'(0)\cdot\frac{\delta(1+\pi)(1-\pi)}{1-\delta}-(1-\pi)\frac{H'(0)}{1-\delta}\cdot\phi\cdot\left(\left(\delta\frac{1+\pi}{2}\right)^2+\frac{1}{\phi^2}\right)\leq-\frac{4}{a\delta}\frac{1}{1-\pi_{DF}}\left(\frac{b}{\delta}+a\right)$$

$$1+\pi-H'(0)\frac{\delta(1+\pi)(1-\pi)}{1-\delta}-(1-\pi)\frac{H'(0)}{1-\delta}\phi\left(\delta^2\frac{(1+\pi)^2}{2}-\frac{1-\delta}{H'(0)}\frac{1+\pi}{1-\pi}\right)\leq-\frac{4}{a\delta}\frac{1}{1-\pi_{DF}}\left(\frac{b}{\delta}+a\right)$$

$$1+\pi-H'(0)\cdot\frac{\delta(1+\pi)(1-\pi)}{1-\delta}-H'(0)\phi\cdot\delta^2\frac{(1-\pi)(1+\pi)^2}{2(1-\delta)}+\phi(1+\pi)\leq-\frac{4}{a\delta}\frac{1}{1-\pi_{DF}}\left(\frac{b}{\delta}+a\right).$$

which can be summarized as

$$1-H'(0)\cdot\frac{\delta(1-\pi)}{1-\delta}-H'(0)\phi\cdot\delta^2\frac{(1-\pi)(1+\pi)}{2(1-\delta)}+\phi\leq-\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)} \tag{A.18}$$

It is convenient to express $H'(0)$ as a function of ϕ using the latter's definition

$$\phi^2=\frac{H'(0)(1-\pi)}{H'(0)(1-\pi)\cdot\left(\delta\frac{1+\pi}{2}\right)^2-(1-\delta)(1+\pi)}$$

$$H'(0)(1-\pi)\cdot\left(\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1\right)=\phi^2(1-\delta)(1+\pi)$$

$$H'(0)(1-\pi)=\frac{\phi^2(1-\delta)(1+\pi)}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}. \tag{A.19}$$

Expressing the sufficient condition (A.18) in terms of ϕ, obtain

$$1-\frac{\delta}{1-\delta}\frac{\phi^2(1-\delta)(1+\pi)}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}-\frac{\phi\delta^2(1+\pi)}{2(1-\delta)}\frac{\phi^2(1-\delta)(1+\pi)}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}+\phi\leq-\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)}$$

$$1-\frac{\phi^2\delta(1+\pi)}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}-\frac{\phi^3\cdot\delta^2\frac{(1+\pi)^2}{2}}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}+\phi\leq-\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)}$$

$$1-\frac{\phi^2\delta(1+\pi)}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}+\frac{-\phi^3\cdot\delta^2\frac{(1+\pi)^2}{4}-\phi}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}\leq-\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)}$$

$$\frac{\phi^2\cdot\delta(1+\pi)+\phi^3\cdot\delta^2\frac{(1+\pi)^2}{4}+\phi}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1}\geq1+\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)} \tag{A.20}$$

By definition $\phi = \frac{\sqrt{H'(0)\cdot\frac{1-\pi}{1+\pi}}}{\sqrt{H'(0)\cdot\delta^2\cdot\frac{1-\pi^2}{4}-(1-\delta)}} \geq \frac{1}{\delta\frac{1+\pi}{2}}$. This implies that sufficient pooling condition (A.20) can be rewritten as

$$\begin{aligned}
\frac{\phi^2\cdot\delta(1+\pi)+\phi^3\cdot\delta^2\frac{(1+\pi)^2}{4}+\phi}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq 1+\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)}\\
\frac{\phi^2\cdot\delta(1+\pi)+\phi\cdot\delta\frac{1+\pi}{2}+\frac{1}{\delta(1+\pi)/2}}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq 1+\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)}\\
\frac{\phi^2\cdot\frac{3}{2}\delta(1+\pi)+\frac{1}{\delta(1+\pi)/2}}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq 1+\frac{4}{a\delta}\frac{b/\delta+a}{(1-\pi_{DF})(1+\pi)}\\
\frac{\phi^2\cdot 3\left(\delta\frac{1+\pi}{2}\right)^2+1}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq \delta\frac{1+\pi}{2}+\frac{2}{a}\frac{b/\delta+a}{1-\pi_{DF}}. \qquad \text{(A.21)}
\end{aligned}$$

Sufficient condition for (A.21) holds for all $\pi \leq \pi_{DF}$ if it holds when the right hand side (A.21) is evaluated at $\pi = \pi_{DF}$. The modified sufficient condition is then given by

$$\begin{aligned}
\frac{\phi^2\cdot 3\left(\delta\frac{1+\pi}{2}\right)^2+1}{\phi^2\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq \frac{d-b}{a}+\frac{2}{a}\frac{b/\delta+a}{2-\frac{2}{\delta}\frac{d-b}{a}}\\
\frac{\phi^2\cdot 3\left(\delta\frac{1+\pi}{2}\right)^2+1}{\phi^2\cdot\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq \frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}. \qquad \text{(A.22)}
\end{aligned}$$

Sufficient condition (A.22) is satisfied if

$$\begin{aligned}
\frac{\phi^2\cdot 3\left(\delta\frac{1+\pi}{2}\right)^2}{\phi^2\cdot\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq \frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}\\
\frac{\phi^2\cdot(1+\pi)}{\phi^2\cdot\left(\delta\frac{1+\pi}{2}\right)^2-1} &\geq \frac{4}{3\delta^2(1+\pi)}\cdot\left(\frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}\right). \qquad \text{(A.23)}
\end{aligned}$$

Condition (A.23) can be interpreted as a lower bound on $H'(0)$ by substituting $H'(0)$ in for ϕ using (A.19)

$$\begin{aligned}
H'(0)(1-\pi) &\geq (1-\delta)\frac{4}{3\delta^2(1+\pi)}\cdot\left(\frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}\right)\\
H'(0) &\geq \frac{4(1-\delta)}{3\delta^2(1-\pi^2)}\cdot\left(\frac{d-b}{a}+\frac{b/\delta+a}{b+a\delta-d}\right).
\end{aligned}$$

As we need this to be satisfied for $\pi \leq \pi_{DF}$ and the right hand side is increasing in π, a sufficient condition is

$$
\begin{aligned}
H'(0) &\geq \frac{4(1-\delta)}{3\delta^2(1-\pi_{DF}^2)} \cdot \left(\frac{d-b}{a} + \frac{b/\delta + a}{b + a\delta - d}\right) \\
H'(0) &\geq \frac{2(1-\delta)}{3\delta} \cdot \frac{1}{1-\pi_{DF}} \cdot \frac{a}{d-b} \cdot \left(\frac{d-b}{a} + \frac{b/\delta + a}{b + a\delta - d}\right) \\
H'(0) &\geq \frac{2(1-\delta)}{3\delta} \cdot \frac{1}{2 - \frac{2}{\delta}\frac{d-b}{a}} \cdot \frac{a}{d-b} \cdot \left(\frac{d-b}{a} + \frac{b/\delta + a}{b + a\delta - d}\right) \\
H'(0) &\geq \frac{1-\delta}{3} \cdot \frac{a}{b + a\delta - d} \cdot \frac{a}{d-b} \cdot \left(\frac{d-b}{a} + \frac{b/\delta + a}{b + a\delta - d}\right).
\end{aligned} \tag{A.24}
$$

A.5 Proof of Proposition 2

Consider an arbitrary sequential stress test $\mathcal{S} = \{S_j, R_j(\cdot)\}_{j=1}^N$. The sequence of signals $\{S_j\}_{j=1}^N$ induces a posterior belief process $(\pi_j)_{j=0}^N$, where π_0 is the initial prior about θ and

$$\pi_j = \mathrm{E}\left[\theta \,|\, S_1, \ldots, S_j\right].$$

Process $(\pi_j)_{j=0}^N$ is a Levy martingale. Moreover, conditional on $\theta = 0$, this process is a super-martingale, while, conditional on $\theta = 1$, it is a sub-martingale, formally stated as

$$\mathrm{E}\left[\pi_j \mid S_1, \ldots, S_{j-1}, \theta = 0\right] \leq \mathrm{E}\left[\pi_j \mid S_1, \ldots, S_{j-1}\right] \stackrel{(i)}{=} \pi_{j-1} \leq \mathrm{E}\left[\pi_j \mid S_1, \ldots, S_{j-1}, \theta = 1\right]. \tag{A.25}$$

By the induction argument, it is sufficient to prove (A.25) for $j = 1$, which follows from an explicit calculation

$$
\begin{aligned}
\mathrm{E}\left[\pi_1 \,|\, \theta = 0\right] - \mathrm{E}\left[\pi_1 \,|\, \theta = 1\right] &= \frac{\mathrm{E}\left[\pi_1 \cdot \mathbb{1}\left\{\theta = 0\right\}\right]}{\mathrm{P}(\theta = 0)} - \frac{\mathrm{E}\left[\pi_1 \cdot \mathbb{1}\left\{\theta = 1\right\}\right]}{\mathrm{P}(\theta = 1)} \\
&= \frac{\mathrm{E}\left[\pi_1 \cdot (1-\theta)\right]}{\mathrm{E}\left[1-\theta\right]} - \frac{\mathrm{E}\left[\pi_1 \cdot \theta\right]}{\mathrm{E}\left[\theta\right]} \\
&= \frac{\mathrm{E}\left[\theta\right] \cdot \mathrm{E}\left[\pi_1 \cdot (1-\theta)\right] - \mathrm{E}\left[1-\theta\right] \cdot \mathrm{E}\left[\pi_1 \cdot \theta\right]}{\mathrm{E}\left[1-\theta\right] \cdot \mathrm{E}\left[\theta\right]} \\
&= \frac{\mathrm{E}\left[\theta\right] \cdot \mathrm{E}\left[\pi_1\right] - \mathrm{E}\left[\pi_1 \cdot \theta\right]}{(1-\pi_0)\pi_0}
\end{aligned}
$$

$$= \frac{\mathrm{E}\left[\mathrm{E}\left[\theta \mid S_1\right]\right] \cdot \mathrm{E}\left[\pi_1\right] - \mathrm{E}\left[\pi_1 \cdot \mathrm{E}\left[\theta \mid S_1\right]\right]}{(1-\pi_0)\pi_0}$$
$$= \frac{\left(\mathrm{E}\left[\pi_1\right]\right)^2 - \mathrm{E}\left[(\pi_1)^2\right]}{(1-\pi_0)\pi_0} \leq 0.$$

Denote by $(B_j, A_j)_{j=0}^N$ the bank's portfolio in the sequential stress test, with $B_0 = b$ and $A_0 = a$. Then, the market value of the bank's portfolio at step j, which we denote by w_j is given by

$$w_j \overset{def}{=} B_j + A_j \cdot \frac{1+\pi_j}{2}.$$

As the bank does not pay dividends before $t = 1$, the bank's portfolio is self-financing, i.e., wealth changes across steps $j-1$ and j are driven by the capital gains/losses on the risky asset position[48]

$$\begin{aligned} w_j &= b_j + a_j \cdot \delta \frac{1+\pi_j}{2} = b_{j-1} + a_{j-1} \cdot \delta \frac{1+\pi_j}{2} \\ &= w_{j-1} + a_{j-1} \cdot \left(\delta \frac{1+\pi_j}{2} - \delta \frac{1+\pi_{j-1}}{2}\right) \\ &= w_0 + \sum_{k=1}^{j} a_{k-1} \cdot \left(\delta \frac{1+\pi_k}{2} - \frac{1+\pi_{k-1}}{2}\right). \end{aligned} \tag{A.26}$$

The belief martingale property (i) in (A.25) implies the martingale property of the bank's wealth process, as follows from (A.26),

$$\begin{aligned} \mathrm{E}\left[w_j \mid S_1, \ldots, S_{j-1}\right] &= \mathrm{E}\left[b_{j-1} + a_{j-1} \cdot \delta \frac{1+\pi_j}{2} \,\middle|\, S_1, \ldots, S_{j-1}\right] \\ &= b_{j-1} + a_{j-1} \cdot \delta \frac{1 + \mathrm{E}\left[\pi_j \mid S_1, \ldots, S_{j-1}\right]}{2} = w_{j-1}. \end{aligned}$$

The same argument implies that, conditional on $\theta = 0$ the wealth process is a super-martingale, while conditional on $\theta = 1$ it is a sub-martingale, just like the belief process in (A.25). It then follows that

$$\mathrm{E}\left[w_N \mid \theta = 0\right] \leq w_0 \leq \mathrm{E}\left[w_N \mid \theta = 1\right],$$

meaning that the expected wealth conditional on $\theta = 0$ ($\theta = 1$) is weakly lower (higher) than the

[48] A simple way to understand the self-financing property is that the bank's budget constraint must be binding in each period.

starting wealth w_0. Moreover, the expected wealth in state $\theta = 0$ is linked to the expected wealth in state $\theta = 1$ via the martingale property

$$\pi_0 \cdot \mathrm{E}\left[w_N \,|\, \theta = 1\right] + (1 - \pi_0) \cdot \mathrm{E}\left[w_N \,|\, \theta = 0\right] = w_0$$
$$\mathrm{E}\left[w_N \,|\, \theta = 0\right] = \frac{w_0 - \pi_0 \cdot \mathrm{E}\left[w_N \,|\, \theta = 1\right]}{1 - \pi_0}. \tag{A.27}$$

Next, notice that any optimal sequential stress test has to end with full disclosure of θ. Otherwise, one can add a fully informative stress test at the end of a sequential test and (weakly) improve welfare by adjusting capital requirements conditional on θ. The expected welfare from the N step sequential stress test $\mathcal{S}$ is given by

$$W(w_0; \pi_0) \overset{def}{=} (1 - \pi_0) \cdot \mathrm{E}\left[V(0, w_N) \,|\, \theta = 0\right] + \pi_0 \cdot \mathrm{E}\left[V(1, w_N) \,|\, \theta = 1\right], \tag{A.28}$$

where $V(\theta, w)$ are defined via (6) in the main text. It follows from Lemma 2 that $V(\theta, w)$ is concave in w. Then, using Jensen's inequality in (A.28) obtain

$$\begin{aligned} W(w_0; \pi_0) &\leq (1 - \pi_0) \cdot V(0, \mathrm{E}\left[w_N \,|\, \theta = 0\right]) + \pi_0 \cdot V(1, \mathrm{E}\left[w_N \,|\, \theta = 1\right]) \\ &\overset{(i)}{=} (1 - \pi_0) \cdot V\left(0, \frac{w_0 - \pi_0 z}{1 - \pi_0}\right) + \pi_0 \cdot V(1, z) \end{aligned}$$

where (i) follows from the martingale property (A.27) of the wealth process and we denoted $z \overset{def}{=} \mathrm{E}\left[w_n \,|\, \theta = 1\right] \geq w_0$. Consider a function $f(z)$ defined as

$$f(z) \overset{def}{=} (1 - \pi_0) \cdot V\left(0, \frac{w_0 - \pi_0 z}{1 - \pi_0}\right) + \pi_0 \cdot V(1, z) \quad \text{ for } z \geq w_0 \geq \delta d,$$

The derivative of $f(z)$ for $z \geq w_0 \geq \delta d$ is given by

$$f'(z) = \pi_0 \cdot \left[\frac{\partial V}{\partial z}(1, z) - \frac{\partial V}{\partial z}(0, \hat{z})\,\Big|_{\hat{z} = \frac{w_0 - \pi_0 z}{1 - \pi_0}}\right] \overset{(i)}{=} \pi_0 \cdot \left[1 + \frac{1 - \delta}{\delta} - \frac{\partial V}{\partial z}(0, \hat{z})\,\Big|_{\hat{z} = \frac{w_0 - \pi_0 z}{1 - \pi_0}}\right] \overset{(i)}{\leq} 0,$$

where equality (i) holds because $\frac{\partial V}{\partial z}(1, z) = 1 + \frac{1-\delta}{\delta}$ for $z \geq \delta \cdot d$ and inequality (ii) holds because $\frac{\partial V}{\partial z}(0, z) \geq 1 + \frac{1-\delta}{\delta}$ due to the positive the marginal default cost in state $\theta = 0$. This implies that

$f'(z) \leq 0$ and, hence, the expected payoff from an arbitrary sequential stress test is bounded above by

$$W(w_0; \pi_0) \leq f(z) \leq f(w_0) = (1 - \pi_0) \cdot V(0, w_0) + \pi_0 \cdot V(1, w_0). \tag{A.29}$$

We conclude the proof by observing that the the upper bound in (A.29) is attained by a stress test implementing a full asset sale by the bank prior to revealing any information about θ, then fully disclosing θ and rebalancing efficiently. Full precautionary recapitalization is, thus, an optimal sequential stress test.

A.6 Proof of Lemma 4

Suppose $H'(0)$ is sufficiently large so that condition (A.15), as well as the additional condition

$$\frac{H'(0)}{1-\delta} \geq \max\left\{\frac{a/(d-b)}{b + a\delta_S - d}, \frac{1}{3}\frac{a^2(d-b)}{b + a\delta_S - d}\left(\frac{d-b}{a} + \frac{b/\delta_S + a}{b + a\delta_S - d}\right), \frac{(1/\delta_B - 2/\delta_S)^2}{1 - \delta_S/\delta_B}\right\} \tag{A.30}$$

Condition (A.30) is weakly stronger than (A.15). A pre-sale of quantity q leads to

$$\pi_{DF}(q) \overset{def}{=} \frac{2}{\delta_S} \cdot \frac{d - b - q \cdot \delta_S(1+\pi)/2}{a - q} - 1. \tag{A.31}$$

Because the right hand side of (A.15) is increasing in π_{DF}, it implies that if it is satisfied at $\pi_{DF}(0)$, then it is satisfied at $\pi_{DF}(q) < \pi_{DF}(0)$. This implies if a default-free stress test was optimal prior to a pre-sale, it is still optimal after a pre-sale. We can, thus, focus on the expected payoffs under a default-free stress test.

The probability $\alpha(q)$ of a bank passing the stress test is given by

$$\begin{aligned}
\alpha(q) &= \frac{\pi + 1 - \frac{2}{\delta_S} \cdot \frac{d-b-q\cdot\delta_S(1+\pi)/2}{a-q}}{2 - \frac{2}{\delta_S} \cdot \frac{d-b-q\cdot\delta_S(1+\pi)/2}{a-q}} \\
&= \frac{(a-q)\cdot\delta_S\frac{1+\pi}{2} + b + q\cdot\delta_S\frac{1+\pi}{2} - d}{(a-q)\cdot\delta_S + b + q\cdot\delta_S\frac{1+\pi}{2} - d} \\
&= \frac{b + a\cdot\delta_S(1+\pi)/2 - d}{b + a\cdot\delta_S - d - q\cdot\delta_S(1-\pi)/2}.
\end{aligned}$$

The expected social welfare from a default-free stress test is

$$v(q) \stackrel{def}{=} \alpha(q)\left(\frac{b+q\cdot\delta_S(1+\pi)/2}{\delta_B}+a-q\right)$$
$$= \frac{b+a\cdot\delta_S(1+\pi)/2-d}{b+a\cdot\delta_S-d-q\cdot\delta_S(1-\pi)/2}\left(\frac{b+q\cdot\delta_S(1+\pi)/2}{\delta_B}+a-q\right)$$

The optimal presale quantity q solves

$$v'(q) = \alpha(q)\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)+\delta_S\frac{1-\pi}{2}\alpha(q)\cdot\frac{\frac{b+q\cdot\delta_S(1+\pi)/2}{\delta_B}+a-q}{b+a\cdot\delta_S-d-q\cdot\delta_S(1-\pi)/2}$$
$$= \alpha(q)\left[\left(\delta_S\frac{1+\pi}{2}-\delta_B\right)+\delta_S\frac{1-\pi}{2}\cdot\frac{b+q\cdot\delta_S(1+\pi)/2+(a-q)\delta_B}{b+a\cdot\delta_S-d-q\cdot\delta_S(1-\pi)/2}\right]$$
$$= \frac{\alpha(q)\delta_S}{b+a\delta_S-d-q\delta_S(1-\pi)/2}\cdot\left[\left(\frac{1+\pi}{2}-\frac{\delta_B}{\delta_S}\right)\left(b+a\cdot\delta_S-d-q\cdot\delta_S\frac{1-\pi}{2}\right)\right.$$
$$\left.+\frac{1-\pi}{2}\cdot\left(b+q\cdot\delta_S\frac{1+\pi}{2}+(a-q)\delta_B\right)\right]$$
$$= \frac{\alpha(q)\delta_S}{b+a\delta_S-d-q\delta_S(1-\pi)/2}\cdot\left[\left(\frac{1+\pi}{2}-\frac{\delta_B}{\delta_S}\right)(b+a\cdot\delta_S-d)+\frac{1-\pi}{2}\cdot(b+(a-q)\delta_B)+q\cdot\delta_B\frac{1-\pi}{2}\right]$$
$$= \frac{\alpha(q)\delta_S}{b+a\delta_S-d-q\delta_S(1-\pi)/2}\cdot\left[\left(\frac{1+\pi}{2}-\frac{\delta_B}{\delta_S}\right)\cdot(b+a\cdot\delta_S-d)+\frac{1-\pi}{2}\cdot(b+a\delta_B)\right]$$
$$= \frac{\alpha(q)\delta_S}{b+a\delta_S-d-q\delta_S(1-\pi)/2}\cdot\left[\left(1-\frac{1-\pi}{2}-\frac{\delta_B}{\delta_S}\right)\cdot(b+a\cdot\delta_S-d)+\frac{1-\pi}{2}\cdot(b+a\delta_B)\right]$$
$$= \frac{\alpha(q)\delta_S}{b+a\delta_S-d-q\delta_S(1-\pi)/2}\cdot\left[\left(1-\frac{\delta_B}{\delta_S}\right)\cdot(b+a\cdot\delta_S-d)+\frac{1-\pi}{2}\cdot(b+a\delta_B-b-a\delta_S+d)\right]$$
$$= \frac{(b+a\delta_S(1+\pi)/2-d)\delta_S}{(b+a\delta_S-d-q\delta_S(1-\pi)/2)^2}\cdot\left[\underbrace{\left(1-\frac{\delta_B}{\delta_S}\right)}_{\leq 0}\cdot(b+a\cdot\delta_S-d)+\frac{1-\pi}{2}\cdot\underbrace{(a\cdot(\delta_B-\delta_S)+d)}_{\geq 0}\right]$$

The above expression is independent of q and, thus, results in a bang-bang solution. The derivative of the above expression with respect to π is equal to $\frac{a(\delta_S-\delta_B)-d}{2}<0$. Thus, there is a threshold $\bar{\pi}$ such that $q(\pi)=q_{DF}(\pi)\cdot\mathbb{1}\{\pi<\bar{\pi}\}$, where

$$\bar{\pi} \stackrel{def}{=} 1+2\cdot\frac{(1-\delta_B/\delta_S)(b-d)+(\delta_S-\delta_B)a}{a\cdot(\delta_B-\delta_S)+d}. \tag{A.32}$$

Denote by $q_{DF}(\pi)$ the quantity of the asset presale that enables a fully informative default-free test in the next period

$$b + q_{DF}(\pi) \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF}(\pi)) \cdot \frac{\delta_S}{2} = d,$$

$$q_{DF}(\pi) \stackrel{def}{=} \frac{d - b - a \cdot \delta_S/2}{\delta_S \pi/2}.$$

Now, suppose that $\pi \leq \bar{\pi}$. It remains to show that it is suboptimal to recapitalize beyond the point in which the subsequent static test is fully informative. Suppose that $q \geq q_{DF}(\pi)$. This implies the subsequent stress test is fully informative. Define

$$\hat{h} \stackrel{def}{=} \frac{1 - \delta - H'(0)\frac{\delta_S}{2}\left(\frac{\delta_S}{2} - \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right)}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} = H'(0) \cdot \left(\frac{\delta_S}{2} - \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right) \stackrel{H'(0)\to\infty}{\to} \frac{1-\delta}{\delta_S}.$$

From Lemma A.2 (below) it follows that the regulator's value function when $\theta = 0$ state is disclosed is dominated by

$$\hat{h} \cdot \left(b + q \cdot \delta_S \frac{1+\pi}{2} + (a - q) \cdot \frac{\delta_S}{2} - d\right).$$

where we have used the fact that $q \geq q_{DF}(\pi)$. The expected welfare from precautionary recapitalizations $q \geq q_{DF}(\pi)$, followed by the optimal stress test which fully discloses θ is given by

$$\begin{aligned}
&\pi \cdot \left(\frac{b + q \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q\right)(1-\delta) + (1-\pi) \cdot \hat{h} \cdot \left(b + q \cdot \delta_S \frac{1+\pi}{2} + (a-q) \cdot \frac{\delta_S}{2} - d\right) \\
=&(q - q_{DF}(\pi)) \cdot \left(\pi\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2} - 1\right)(1-\delta) + (1-\pi)\hat{h} \cdot \delta_S \frac{\pi}{2}\right) \\
&+\pi \cdot \left(\frac{b + q_{DF}(\pi) \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q_{DF}(\pi)\right)(1-\delta) \\
&+(1-\pi) \cdot \hat{h} \cdot \left(b + q_{DF}(\pi) \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF}(\pi)) \cdot \frac{\delta_S}{2} - d\right) \\
\stackrel{(i)}{=}&(q - q_{DF}(\pi))\frac{(1-\pi)\pi}{2}(\hat{h} \cdot \delta_S - (1-\delta)) \\
&+(q - q_{DF}(\pi))\left(\pi\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2} - 1\right)(1-\delta) + (1-\pi)(1-\delta)\frac{\pi}{2}\right) \\
&+\pi \cdot \left(\frac{b + q_{DF}(\pi) \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q_{DF}(\pi)\right)(1-\delta) \\
&+(1-\pi) \cdot \hat{h} \cdot \left(b + q_{DF}(\pi) \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF}(\pi)) \cdot \frac{\delta_S}{2} - d\right)
\end{aligned}$$

$$
\begin{aligned}
=&(q - q_{DF}(\pi)) \cdot \frac{(1-\pi)\pi}{2} \cdot \left(\hat{h} \cdot \delta_S - (1-\delta)\right) + (q - q_{DF}(\pi)) \cdot (1-\delta)\pi \left(\frac{\delta_S}{\delta_B} - 1\right) \cdot \frac{1+\pi}{2} \\
&+\pi \cdot \left(\frac{b + q_{DF}(\pi) \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q_{DF}(\pi)\right)(1-\delta) \\
&+(1-\pi) \cdot \hat{h} \cdot \left(b + q_{DF}(\pi) \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF}(\pi)) \cdot \frac{\delta_S}{2} - d\right) \\
=&(q - q_{DF}(\pi)) \cdot \frac{\pi}{2} \cdot \left((1-\pi)\left(\hat{h} \cdot \delta_S - (1-\delta)\right) + (1-\delta)(1+\pi)\left(\frac{\delta_S}{\delta_B} - 1\right)\right) \\
&+\pi \cdot \left(\frac{b + q_{DF}(\pi) \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q_{DF}(\pi)\right)(1-\delta) \\
&+(1-\pi) \cdot \hat{h} \cdot \left(b + q_{DF}(\pi) \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF}(\pi)) \cdot \frac{\delta_S}{2} - d\right) \\
\overset{(i)}{\leq}&\pi \cdot \left(\frac{b + q_{DF}(\pi) \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q_{DF}(\pi)\right)(1-\delta) \qquad \text{(A.33)} \\
&+(1-\pi) \cdot \hat{h} \cdot \left(b + q_{DF}(\pi) \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF}(\pi)) \cdot \frac{\delta_S}{2} - d\right) \\
=& \pi \cdot \left(\frac{b + q_{DF}(\pi) \cdot \delta_S(1+\pi)/2}{\delta_B} + a - q_{DF}(\pi)\right)(1-\delta) \qquad \text{(A.34)}
\end{aligned}
$$

which shows that, whenever inequality (i) in (A.33) is satisfied, then precautionary recapitalization $q_{DF}(\pi)$ dominates $q > q_{DF}(\pi)$ as (A.34) is the expected welfare from precautionary recapitalization $q_{DF}(\pi)$ followed by full information disclosure. Inequality (i) in (A.33) holds whenever $H'(0)$ is sufficiently large so that

$$
(1-\pi)\left(\hat{h} \cdot \delta_S - (1-\delta)\right) + (1-\delta)(1+\pi)\left(\frac{\delta_S}{\delta_B} - 1\right) \leq 0.
$$

A sufficient condition for the above to hold is if it holds at $\pi = 0$. Thus

$$
\begin{aligned}
\hat{h} \cdot \delta_S - (1-\delta) + (1-\delta)\left(\frac{\delta_S}{\delta_B} - 1\right) &\leq 0 \\
H'(0) \cdot \left(\frac{\delta_S}{2} - \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right) \cdot \delta_S - (1-\delta) + (1-\delta)\left(\frac{\delta_S}{\delta_B} - 1\right) &\leq 0 \\
H'(0) \cdot \frac{\delta_S^2}{2} - (1-\delta) + (1-\delta)\left(\frac{\delta_S}{\delta_B} - 1\right) &\leq H'(0)\delta_S\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}} \\
H'(0) \cdot \frac{\delta_S}{2} + (1-\delta)\left(\frac{1}{\delta_B} - \frac{2}{\delta_S}\right) &\leq H'(0)\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}
\end{aligned}
$$

$$H'(0)^2 \cdot \frac{\delta_S^2}{4} + H'(0)\delta_S(1-\delta)\left(\frac{1}{\delta_B} - \frac{2}{\delta_S}\right) + (1-\delta)^2\left(\frac{1}{\delta_B} - \frac{2}{\delta_S}\right)^2 \le H'(0)^2 \cdot \frac{\delta_S^2}{4} - H'(0)\cdot(1-\delta)$$
$$H'(0)\delta_S(1-\delta)\left(\frac{1}{\delta_B} - \frac{1}{\delta_S}\right) + (1-\delta)^2\left(\frac{1}{\delta_B} - \frac{2}{\delta_S}\right)^2 \le 0$$
$$H'(0)\delta_S\left(\frac{1}{\delta_B} - \frac{1}{\delta_S}\right) + (1-\delta)\left(\frac{1}{\delta_B} - \frac{2}{\delta_S}\right)^2 \le 0$$
$$\frac{(1-\delta)\cdot(1/\delta_B - 2/\delta_S)^2}{1-\delta_S/\delta_B} \le H'(0).$$

A.7 Proof of Proposition 3

Suppose

$$H'(0) \ge \max\left\{\frac{(1-\delta)\cdot(1/\delta_B - 2/\delta_S)^2}{1-\delta_S/\delta_B}, \quad \frac{1-\delta}{\delta_S}\frac{1}{d-b}\left[\frac{d-b}{\delta_S} + \frac{a}{2}\frac{b/\delta_B + a}{b + a\delta_S - d} - \frac{b + a\delta_S/2}{\delta_B}\right]\right\}. \tag{A.35}$$

Denote by $u_0(b,a)$ to be the expected social welfare in state $\theta = 0$, given by

$$u_0(b,a) = \max_{\hat{q}_B, \hat{q}_S}\left\{(a + \hat{q}_B - \hat{q}_S)\frac{1-\delta}{2} + \mathrm{E}\left[H\left(b + \frac{\hat{q}_S\delta_S - \hat{q}_B\delta_B}{2} + (a + \hat{q}_B - \hat{q}_S)X - d\right)\right]\right\} \tag{A.36}$$

subject to $\hat{q}_B, \hat{q}_S \ge 0$, a no short-selling constraint $a + \hat{q}_B - \hat{q}_S \ge 0$, and the budget constraint

$$b + \frac{\hat{q}_S\delta_S - \hat{q}_B\delta_B}{2} \ge 0. \tag{A.37}$$

Similarly, denote by $u_1(b,a)$ to be the expected social welfare in state $\theta = 0$, given by in closed form as

$$u_1(b,a) \overset{def}{=} \left(\frac{b}{\delta_B} + a\right)\cdot(1-\delta) + H\left(\frac{b}{\delta_B} + a - d\right).$$

Proposition 3, part 1: expected welfare upper bounds

Lemma A.2. *The bank's terminal value from portfolio (b,a), previously denoted by $u_0(b,a)$, is dominated by*

$$\bar{u}_0(b,a) \overset{def}{=} \begin{cases} \left(b + a\cdot\frac{\delta_S}{2} - d\right) \dfrac{1-\delta - H'(0)\frac{\delta_S}{2}\left(\frac{\delta_S}{2} - \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right)}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} & \text{if } b + a\cdot\frac{\delta_S}{2} \geq d, \\ \left(b + a\cdot\frac{\delta_S}{2} - d\right) \dfrac{H'(0)\frac{\delta_S}{2}\left(\frac{\delta_S}{2} + \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right) - 1 + \delta}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} & \text{if } b + a\cdot\frac{\delta_S}{2} < d. \end{cases} \tag{A.38}$$

Proof. The solution to (A.36) subject to (A.37) is dominated by the solution to the unconstrained problem (A.36) in which (A.37) is not required and the bank can purchase the asset at the same discount δ_S as it can sell it. The latter allows to net the trade transactions by defining $\hat{q} = \hat{q}_B - \hat{q}_S$. The solution $u_0(b,a)$ is, thus, weakly lower than $\bar{u}_0(b,a)$ defined as

$$\bar{u}_0(b,a) \overset{def}{=} \max_{\hat{q}} \left\{ (a+\hat{q})\frac{1-\delta}{2} + \mathrm{E}\left[H\left(b - \hat{q}\cdot\frac{\delta_S}{2} + (a+\hat{q})\cdot X - d\right)\right]\right\} \tag{A.39}$$

subject to no short selling $a + q \geq 0$. Direct calculation, similar to Lemma A.21 yields that $\bar{u}_0(b,a)$ is given by (A.38). □

It is convenient to define $\hat{\delta}$ as

$$\frac{1-\hat{\delta}}{\delta_S} \overset{def}{=} \frac{1-\delta - H'(0)\cdot\frac{\delta_S}{2}\cdot\left(\frac{\delta_S}{2} - \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right)}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H'(0)}}} = H'(0)\cdot\left(\frac{\delta_S}{2} - \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right). \tag{A.40}$$

It is convenient to define $\hat{\delta}$ as

$$\begin{aligned} \hat{h} &\overset{def}{=} \frac{H'(0)\frac{\delta_S}{2}\left(\frac{\delta_S}{2} + \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}\right) - 1 + \delta}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} - \frac{1-\hat{\delta}}{\delta_S} \\ &= \frac{H'(0)\frac{\delta_S^2}{2} - 2(1-\delta)}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} = H'(0)\cdot 2\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}} < H'(0). \end{aligned} \tag{A.41}$$

Using this notation, can rewrite the upper bound (A.38) as

$$\bar{u}_0(b,a) = \left(b + a \cdot \frac{\delta_S}{2} - d\right) \cdot \frac{1-\hat{\delta}}{\delta_S} + \left[b + a \cdot \frac{\delta_S}{2} - d\right]^- \cdot \hat{h}.$$

In the relaxed problem, the distress cost is smaller, captured by $\hat{h} \cdot \min(x,0) > H(x)$ smaller, but the benefit of holding the risky asset is higher in state $\theta = 0$.

Corollary A.1. *The expected value of the bank in state $\theta = 1$ is bounded above by*

$$u_1(b,a) \leq \bar{u}_1(b,a) \overset{def}{=} \frac{b + a\delta_B}{\delta_B} \cdot (1-\delta).$$

Proof. We are simply removing the default cost $H(b/\delta_B + a - d) \leq 0$. □

The expected social welfare of rebalancing the bank's portfolio from the starting portfolio (b,a) prior to disclosing θ fully and subsequently rebalancing is bounded above by

$$\begin{aligned} \bar{U}(\pi; b, a) \overset{def}{=} \max_{\hat{q}_B, \hat{q}_A} \Bigg\{ & \pi \cdot \bar{u}_1 \left(b + (\hat{q}_S \delta_S - \hat{q}_B \delta_B) \cdot \frac{1+\pi}{2}, \quad a + \hat{q}_B - \hat{q}_S \right) \\ & + (1-\pi) \cdot \bar{u}_0 \left(b + (\hat{q}_S \delta_S - \hat{q}_B \delta_B) \cdot \frac{1+\pi}{2}, \quad a + \hat{q}_B - \hat{q}_S \right) \Bigg\} \end{aligned} \tag{A.42}$$

subject to $\hat{q}_B, \hat{q}_S \geq 0$, no short sales $a + \hat{q}_B - \hat{q}_S \geq 0$, and the budget constraint. In what follows, we identify properties of $\bar{U}(\pi; b, a)$.

Proposition 3, part 2: suboptimality of asset pre-purchases in the relaxed problem.

Lemma A.3 (No pre-purchases)**.** *The optimal pre-purchase quantity is $q_B = 0$.*

Proof. Suppose, from the contrary, the optimal $q_B > 0$. Due to the difference in discounts $\delta_B \geq \delta_S$, it follows that $q_S = 0$. This implies that

$$q_B \in \arg\max_{\hat{q}} \left\{ \pi \cdot \bar{u}_1 \left(b - \hat{q}\delta_B \cdot \frac{1+\pi}{2},\ a + \hat{q} \right) + (1-\pi) \cdot \bar{u}_0 \left(b - \hat{q}\delta_B \cdot \frac{1+\pi}{2},\ a + \hat{q} \right) \right\}.$$

The marginal value of a marginal pre-purchase increase is given by

$$
\begin{aligned}
&\pi \cdot \left[-\frac{\partial \bar{u}_1}{\partial b}\left(b - q_B\delta_B \cdot \frac{1+\pi}{2},\ a+q_B\right)\cdot \delta_B\frac{1+\pi}{2} + \frac{\partial \bar{u}_1}{\partial a}\left(b - q_B\delta_B \cdot \frac{1+\pi}{2},\ a+q_B\right)\right] \\
&+(1-\pi)\cdot \left[-\frac{\partial \bar{u}_0}{\partial b}\left(b - q_B\delta_B \cdot \frac{1+\pi}{2},\ a+q_B\right)\cdot \delta_B\frac{1+\pi}{2} + \frac{\partial \bar{u}_0}{\partial a}\left(b - q_B\delta_B \cdot \frac{1+\pi}{2},\ a+q_B\right)\right] \\
&=\pi \cdot \left(-\frac{1-\delta}{\delta_B}\cdot \delta_B\frac{1+\pi}{2} + 1-\delta\right) \\
&+(1-\pi)\cdot \left(-\frac{1-\hat{\delta}}{\delta_S} - \hat{h}\cdot \mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\}\right)\cdot \delta_B\frac{1+\pi}{2} \\
&+(1-\pi)\cdot \left(\frac{1-\hat{\delta}}{2} + \hat{h}\frac{\delta_S}{2}\cdot \mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\}\right) \\
&=\pi\cdot(1-\delta)\cdot\frac{1-\pi}{2} \\
&+(1-\pi)\cdot\left(\frac{1-\hat{\delta}}{2}\left(1-\frac{\delta_B}{\delta_S}(1+\pi)\right) + \hat{h}\cdot\left(\frac{\delta_S}{2} - \delta_B\frac{1+\pi}{2}\right)\mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\}\right) \\
&\overset{(i)}{\leq}(1-\hat{\delta})\frac{1-\pi}{2}\left(\pi + 1 - \frac{\delta_B}{\delta_S}(1+\pi)\right) \\
&+\hat{h}\cdot\frac{1-\pi}{2}\Big(\delta_S - \delta_B(1+\pi)\Big)\cdot \mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\} \\
&\overset{(ii)}{\leq}(1-\hat{\delta})\frac{1-\pi}{2}(1+\pi)\left(1-\frac{\delta_B}{\delta_S}\right) \\
&+\hat{h}\cdot\frac{1-\pi}{2}\Big(\pi\delta_S + \delta_S - \delta_B(1+\pi)\Big)\cdot \mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\} \\
&=(1+\pi)\left(1-\frac{\delta_B}{\delta_S}\right)\frac{1-\pi}{2}\left(1-\hat{\delta} + \delta_S\hat{h}\cdot \mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\}\right) \leq 0.
\end{aligned}
$$

Inequality (i) follows from the fact that $1-\delta < 1-\hat{\delta}$, and (ii) holds because $\hat{h}(1-\pi)\cdot\mathbb{1}\{\ldots\} \geq 0$. □

Lemma A.3 guarantees that the optimal $q_B = 0$ in (A.39). For notational simplicity it is then convenient to drop the S subscript to denote $\hat{q} = \hat{q}_S$ and $q = q_S$. It is convenient to define $q_{DF}(\pi)$ as the minimum size of precautionary recapitalization such that a full-disclosure default-free test is feasible. Formally,

$$
q_{DF}(\pi) \overset{def}{=} \min\left\{\hat{q} \in [0,a]: \qquad b + q_{DF}(\pi)\cdot\delta_S\frac{1+\pi}{2} + (a - q_{DF}(\pi))\cdot\frac{\delta_S}{2} \geq d\right\}
$$

Direct calculation yields

$$q_{DF}(\pi) = \begin{cases} \frac{d - a\cdot\delta_S/2 - b}{\delta_S \pi/2}, & \text{if } b + a\cdot\delta_S \frac{1+\pi}{2} \geq d, \\ a, & \text{if } b + a\cdot\delta_S\frac{1+\pi}{2} < d. \end{cases} \tag{A.43}$$

Proposition 3, part 3: optimal precautionary recapitalization in the relaxed problem.

Lemma A.4 (Optimal pre-sales)**.** *There exists a threshold $\bar{\pi} \in [0, 1)$ given by*

$$\bar{\pi} = 1 - 2(1-\delta)\cdot \frac{1 - \delta_S/\delta_B}{\hat{h}\cdot\delta_S - (1-\delta)\cdot\delta_S/\delta_B + 1 - \hat{\delta}}.$$

such that if $\bar{\pi} > \pi_{DF}$, then

$$q(\pi) = \begin{cases} 0, & \textit{if } \pi \geq \bar{\pi}, \\ \frac{d - a\cdot\delta_S/2 - b}{\delta_S\pi/2}, & \textit{if } \pi \in [\pi_{DF}, \bar{\pi}], \\ a, & \textit{if } \pi \leq \pi_{DF}, \end{cases} \tag{A.44}$$

and if $\bar{\pi} < \pi_{DF}$, then

$$q(\pi) = \begin{cases} 0, & \textit{if } \pi \geq \bar{\pi}, \\ a, & \textit{if } \pi \leq \bar{\pi}. \end{cases} \tag{A.45}$$

Proof. The marginal benefit of increasing presale quantity from level q is given by

$$\begin{aligned} &\pi\cdot\left[\frac{\partial \bar{u}_1}{\partial b}\left(b + q\delta_S\cdot\frac{1+\pi}{2}, a-q\right)\cdot\delta_S\frac{1+\pi}{2} - \frac{\partial\bar{u}_1}{\partial b}\left(b+q\delta_S\cdot\frac{1+\pi}{2}, a-q\right)\right] \\ &+(1-\pi)\cdot\left[\frac{\partial\bar{u}_0}{\partial b}\left(b+q\delta_S\cdot\frac{1+\pi}{2}, a-q\right)\cdot\delta_S\frac{1+\pi}{2} - \frac{\partial\bar{u}_0}{\partial a}\left(b+q\delta_S\cdot\frac{1+\pi}{2}, a-q\right)\right] \\ &= \pi\cdot\left(\frac{1-\delta}{\delta_B}\cdot\delta_S\frac{1+\pi}{2} - (1-\delta)\right) \\ &+(1-\pi)\cdot\left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h}\cdot\mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\}\right)\cdot\delta_S\frac{1+\pi}{2} \\ &-(1-\pi)\cdot\left(\frac{1-\hat{\delta}}{2} + \hat{h}\frac{\delta_S}{2}\cdot\mathbb{1}\left\{b - q_B\delta_B\frac{1+\pi}{2} + (a+q_B)\frac{\delta_S}{2} < d\right\}\right) \end{aligned}$$

$$=\pi\cdot(1-\delta)\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)+(1-\pi)\cdot\left[\frac{(1-\hat{\delta})\pi}{2}+\hat{h}\frac{\delta_S\pi}{2}\mathbb{1}\left\{b-q_B\delta_B\frac{1+\pi}{2}+(a+q_B)\frac{\delta_S}{2}<d\right\}\right] \quad \text{(A.46)}$$

Consider two cases:

1. $\theta = 0$ bank is solvent, captured by

$$d < b + q\delta_S\frac{1+\pi}{2} + (a-q)\frac{\delta_S}{2}.$$

Then, the marginal value of a presale in (A.46) can be expressed as

$$\pi(1-\delta)\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)+(1-\pi)\frac{(1-\hat{\delta})\pi}{2} \vee 0$$

$$(1-\delta)\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)+(1-\pi)\frac{(1-\hat{\delta})}{2} \vee 0$$

$$\pi\cdot\underbrace{\left(\frac{1-\delta}{2}\frac{\delta_S}{\delta_B}-\frac{1-\hat{\delta}}{2}\right)}_{<0}+\frac{1-\delta}{2}\left(\frac{\delta_S}{\delta_B}-2\right)+\frac{1-\hat{\delta}}{2}\overset{(i)}{\leq}0,$$

where (i) holds when $\frac{1-\hat{\delta}}{2}$ is close enough to $\frac{1-\delta}{2}$ such that

$$\frac{1-\hat{\delta}}{2}\leq\frac{1-\delta}{2}\left(2-\frac{\delta_S}{\delta_B}\right),$$

$$H'(0)\cdot\left(\frac{\delta_S}{2}-\sqrt{\frac{\delta_S^2}{4}-\frac{1-\delta}{H'(0)}}\right)\leq\frac{1-\delta}{2}\delta_S\left(2-\frac{\delta_S}{\delta_B}\right), \quad \text{(A.47)}$$

$$H'(0)\geq\frac{(1-\delta)\cdot(1/\delta_B-2/\delta_S)^2}{1-\delta_S/\delta_B}. \quad \text{(A.48)}$$

where (A.47) follows from the definition of (A.40) and (A.48) follows from the final derivations of the proof in Lemma 4.

2. $\theta = 0$ bank is in distress, captured by

$$b + q\delta_S\frac{1+\pi}{2} + (a-q)\frac{\delta_S}{2} < d. \quad \text{(A.49)}$$

Then, the marginal value of a presale in (A.46) can be expressed as

$$\pi(1-\delta)\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)+(1-\pi)\frac{(1-\hat{\delta})\pi}{2}+(1-\pi)\hat{h}\frac{\delta_S\pi}{2}\vee 0$$
$$(1-\delta)\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)+(1-\pi)\frac{1-\hat{\delta}}{2}+(1-\pi)\hat{h}\frac{\delta_S}{2}\vee 0$$
$$\pi\left(\frac{1-\delta}{2}\frac{\delta_S}{\delta_B}-\frac{1-\hat{\delta}}{2}-\hat{h}\frac{\delta_S}{2}\right)+\frac{1-\delta}{2}\left(\frac{\delta_S}{\delta_B}-2\right)+\frac{1-\hat{\delta}}{2}+\hat{h}\frac{\delta_S}{2}\vee 0.$$

Denote by $\bar{\pi}$ the cutoff belief

$$\bar{\pi}\overset{def}{=}\frac{\frac{1-\delta}{2}\left(\frac{\delta_S}{\delta_B}-2\right)+\frac{1-\hat{\delta}}{2}+\hat{h}\frac{\delta_S}{2}}{-\frac{1-\delta}{2}\frac{\delta_S}{\delta_B}+\frac{1-\hat{\delta}}{2}+\hat{h}\frac{\delta_S}{2}}=1-2(1-\delta)\cdot\frac{1-\delta_S/\delta_B}{\hat{h}\cdot\delta_S-(1-\delta)\cdot\delta_S/\delta_B+1-\hat{\delta}}$$
$$1-\bar{\pi}=2(1-\delta)\cdot\frac{1-\delta_S/\delta_B}{\hat{h}\cdot\delta_S-(1-\delta)\cdot\delta_S/\delta_B+1-\hat{\delta}}=2(1-\delta)\cdot\left(\frac{1}{\delta_S}-\frac{1}{\delta_B}\right)\cdot\frac{1}{\hat{h}}+O\left(\frac{1}{\hat{h}^2}\right).$$

When $\pi>\bar{\pi}$ then presale is suboptimal despite the distress of the bank in $\theta=0$ state. When $\pi<\bar{\pi}$ marginal presale is optimal, and it remains optimal whenever (A.49) is strict. As a result, for $\pi<\bar{\pi}$ presale makes inequality (A.49) bind, which is possible for $\pi>\pi_{DF}$ or presale remains optimal for any q, resulting in the corner solution of $q=a$, which only happens for $\pi\le\pi_{DF}$.

□

Proposition 3, part 4: optimality of persuation over $[\pi^*,1]$ prior to recapitalization.

Denote by $\text{cav}_1[f](x,y)$ to be the concave envelope of function $f(x,y)$ along dimension x. Formally, $\text{cav}_1[f](x,y)$ is the smallest function, concave in the first variable, which exceeds $f(x,y)$ for every (x,y). Denote by $\bar{U}^*(\pi;b,a)$ to be the concave hull of $\bar{U}(\pi;b,a)$ given by

$$\bar{U}^*(\pi;b,a)\overset{def}{=}\text{cav}_1\left[\bar{U}\right](\pi;b,a). \tag{A.50}$$

Lemma A.5 (Concavification)**.** *The value function $\bar{U}(\pi)$ to the relaxed problem is concave for $\pi<\min(\pi_{DF},\bar{\pi})$ and (weakly) convex for $\pi>\min(\pi_{DF},\bar{\pi})$ with a positive kink at $\bar{\pi}$. Hence, there* *exists a* $\pi^* \leq \min(\pi_{DF}, \bar{\pi})$ *such that*

$$\bar{U}^*(\pi; b, a) = \begin{cases} \dfrac{1-\pi}{1-\pi^*} \cdot U(\pi^*; b, a) + \dfrac{\pi - \pi^*}{1-\pi^*} \cdot U(1; b, a) & \text{if} \quad \pi \geq \pi^*, \\ U(\pi; b, a) & \text{if} \quad \pi \leq \pi^*. \end{cases}$$

Proof. Lemma A.3 shows that $q_B(\pi) \equiv 0$ and Lemma A.4 shows that $q_S(\pi) = q_{DF}(\pi)\mathbb{1}\{\pi < \bar{\pi}\}$ where $\bar{\pi}$ is determined by the size of trade frictions (δ_B/δ_S) and marginal default cost $\hat{h}$.

For $\pi \geq \bar{\pi}$ the value function $\bar{U}$ is given by

$$\bar{U}(\pi; b, a) = \pi \frac{1-\delta}{\delta_B}(b + a\delta_B) + (1-\pi)\left(b + a\delta_S \frac{1}{2} - d\right) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h}\right).$$

In this region the value function is linear (hence weakly convex) and the derivative is given by

$$\frac{\partial \bar{U}}{\partial \pi}(\pi; b, a) = \frac{1-\delta}{\delta_B}(b + a\delta_B) - \left(b + a\delta_S \frac{1}{2} - d\right) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h}\right).$$

For $\pi \in (\pi_{DF}, \bar{\pi})$ optimal presale is $q_{DF} = \frac{d-b-a\delta_S/2}{\delta_S \pi/2}$ and the value function is given by

$$\begin{aligned} \bar{U}(\pi; b, a) &= \pi \frac{1-\delta}{\delta_B}\left(b + q_{DF} \cdot \delta_S \frac{1+\pi}{2} + (a - q_{DF})\delta_B\right) + (1-\pi) \cdot 0 \\ &= \pi \frac{1-\delta}{\delta_B}\left(b + a\delta_B + q_{DF} \cdot \left(\delta_S \frac{1+\pi}{2} - \delta_B\right)\right) \\ &= \frac{1-\delta}{\delta_B}\left(\pi(b + a\delta_B) + \pi q_{DF} \cdot \left(\delta_S \frac{1+\pi}{2} - \delta_B\right)\right) \end{aligned}$$

since $\pi q_{DF} = \frac{d-b-a\delta_S/2}{\delta_S/2}$ the value function above is linear in π. In this region the derivative is given by

$$\begin{aligned} \frac{\partial \bar{U}}{\partial \pi}(\pi; b, a) &= \frac{1-\delta}{\delta_B}\left(b + a\delta_B + \pi q_{DF} \cdot \delta_S \frac{1}{2}\right) = \frac{1-\delta}{\delta_B}\left(b + a\delta_B + d - b - a\delta_S \frac{1}{2}\right) \\ &= \frac{1-\delta}{\delta_B}(b + a\delta_B) - \frac{1-\delta}{\delta_B}\left(b + a\delta_S \frac{1}{2} - d\right). \end{aligned}$$

Notice that

$$\frac{1-\delta}{\delta_B} \leq \frac{1-\delta}{\delta_S} < \frac{1-\hat{\delta}}{\delta_S} < \frac{1-\hat{\delta}}{\delta_S} + \hat{h},$$

hence, whenever $\bar{\pi} > \pi_{DF}$ we have $U'(\bar{\pi}-) < U'(\bar{\pi}+)$. As a result, $U(\pi)$ is strictly convex in $\pi \in [\pi_{DF}, 1]$.

Consider $\pi < \min(\pi_{DF}, \bar{\pi})$. The expected value from the full presale is

$$
\begin{aligned}
\bar{U}(\pi; b, a) &= \pi \cdot \bar{u}_1 \left(b + a \cdot \delta_S \frac{1+\pi}{2}, 0 \right) + (1-\pi) \cdot \bar{u}_0 \left(b + a \cdot \delta_S \frac{1+\pi}{2}, 0 \right). \\
&= \pi \cdot \frac{b + a \cdot \delta_S \frac{1+\pi}{2}}{\delta_B} \cdot (1-\delta) + (1-\pi) \cdot \left(b + a \cdot \delta_S \frac{1+\pi}{2} - d \right) \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) \\
&= \pi^2 \cdot a \left((1-\delta) \frac{\delta_S}{2\delta_B} - \frac{\delta_S}{2} \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) \right) + \pi \left((1-\delta) \frac{b + a\delta_S/2}{\delta_B} + (d-b) \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) \right) \\
&+ \left(b + a \cdot \frac{\delta_S}{2} - d \right) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right).
\end{aligned}
$$

It is convenient to express for $\pi < \pi_{DF}$

$$
\begin{aligned}
\frac{\partial \bar{U}}{\partial \pi}(\pi; b, a) &= 2\pi a \left((1-\delta) \frac{\delta_S}{2\delta_B} - \frac{\delta_S}{2} \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) \right) + \left((1-\delta) \frac{b + a\delta_S/2}{\delta_B} + (d-b) \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) \right) \\
&= \pi \cdot a \left((1-\delta) \frac{\delta_S}{\delta_B} - \left(1 - \hat{\delta} + \hat{h}\delta_S \right) \right) + (1-\delta) \cdot \frac{b + a\delta_S/2}{\delta_B} + (d-b) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right).
\end{aligned}
\tag{A.51}
$$

We can see that as long as

$$
\begin{aligned}
(1-\delta) \frac{\delta_S}{\delta_B} - \left(1 - \hat{\delta} + \hat{h}\delta_S \right) &\leq 0, \\
(1-\delta) \frac{\delta_S}{\delta_B} - \delta_S \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) &\leq 0, \\
\frac{1-\delta}{\delta_B} - \frac{H'(0) \frac{\delta_S}{2} \left(\frac{\delta_S}{2} + \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}} \right) - 1 + \delta}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} &\leq 0, \\
\frac{1-\delta}{\delta_B} - H'(0) \cdot \frac{\delta_S}{2} - \frac{H'(0) \frac{\delta_S^2}{4} - (1-\delta)}{\sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}}} &\leq 0, \\
\frac{1-\delta}{\delta_B} - H'(0) \cdot \frac{\delta_S}{2} - H'(0) \sqrt{\frac{\delta_S^2}{4} - \frac{1-\delta}{H'(0)}} &\leq 0, \\
\frac{(1-\delta)^2}{\delta_B^2} - H'(0) \frac{\delta_S}{\delta_B} (1-\delta) &\leq -H'(0) \cdot (1-\delta)
\end{aligned}
$$

which is always satisfied whenever $H'(0) \geq \frac{(1-\delta)\cdot(1/\delta_B - 2/\delta_S)^2}{1-\delta_S/\delta_B}$, which we have assumed is satisfied in the statement of the Proposition. In this case, $\frac{\partial \bar{U}}{\partial \pi}(\pi, b, a)$ is decreasing in π, implying that $\bar{U}(\pi)$ is locally concave in π for $\pi < \min(\pi_{DF}, \bar{\pi})$. □

Denote by $\pi^* \leq \min(\pi_{DF}, \bar{\pi})$ the optimal concavification point. The objective $\bar{U}$ is strictly concave for $\pi < \pi^*$ and for $\pi > \pi^*$ it is given by the convex combination.

Lemma A.6. *If $\pi^* < \pi_{DF}$, then it satisfies the quadratic equation*

$$2 - \frac{\delta_S}{\delta_B}(1+\pi^*) = (1-\pi^*)^2 \left(\frac{1-\hat{\delta}}{1-\delta} + \frac{\hat{h}\delta_S}{1-\delta} \right) + \pi^*(1-\pi^*)\frac{\delta_S}{\delta_B}. \tag{A.52}$$

Proof. The expected utilities are given by

$$\bar{U}(1; b, a) = \left(\frac{b}{\delta_B} + a \right)(1-\delta).$$

Suppose $\pi < \min(\pi_{DF}, \bar{\pi})$. The expected utility to the regulator is derived from the full asset presale and given by

$$\bar{U}(\pi; b, a) = \pi \cdot \frac{b + a \cdot \delta_S(1+\pi)/2}{\delta_B} \cdot (1-\delta) + (1-\pi) \cdot \left(b + a \cdot \delta_S \frac{1+\pi}{2} - d \right) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right).$$

The derivative with respect to π for $\pi < \min(\pi_{DF}, \bar{\pi})$ is given by

$$\begin{aligned}
\frac{\partial \bar{U}}{\partial \pi}(\pi; b, a) &= \frac{b + a \cdot \delta_S(1+\pi)/2}{\delta_B} \cdot (1-\delta) + \pi \cdot a \cdot \frac{\delta_S}{2\delta_B} \cdot (1-\delta) \\
&\quad - \left(b + a \cdot \delta_S \frac{1+\pi}{2} - d \right) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) + (1-\pi) \cdot a \cdot \frac{\delta_S}{2} \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right) \\
&= \pi \cdot a \cdot \left(\frac{\delta_S}{\delta_B}(1-\delta) - \left(1 - \hat{\delta} + \hat{h}\delta_S \right) \right) + \frac{b + a \cdot \delta_S/2}{\delta_B} \cdot (1-\delta) + (d-b) \cdot \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right).
\end{aligned}$$

The tangency condition for π^*, given by $\frac{U(1)-U(\pi^*)}{1-\pi^*} = U'(\pi^*)$ is then written as

$$\frac{\left(\frac{b}{\delta_B} + a \right)(1-\delta) - \pi^* \cdot \frac{b + a\cdot\delta_S(1+\pi^*)/2}{\delta_B}(1-\delta) - (1-\pi^*) \cdot \left(b + a\delta_S \frac{1+\pi^*}{2} - d \right) \left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h} \right)}{1-\pi^*}$$

$$=\pi \cdot a \cdot \left((1-\delta)\frac{\delta_S}{\delta_B} - \left(1-\hat{\delta}+\hat{h}\delta_S\right)\right) + (1-\delta)\cdot \frac{b+a\delta_S/2}{\delta_B} + (d-b)\cdot \left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right).$$

Simplifying terms obtain

$$\frac{\left(\frac{b}{\delta_B}+a\right)(1-\delta) - \pi^* \frac{b+a\delta_S\frac{1+\pi^*}{2}}{\delta_B}(1-\delta)}{1-\pi^*} - a\delta_S\frac{1+\pi^*}{2}\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)$$
$$=\pi^* a\left((1-\delta)\frac{\delta_S}{\delta_B} - \delta_S\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)\right) + (1-\delta)\frac{b+a\delta_S/2}{\delta_B}.$$

Simplify terms further obtain

$$\frac{\left(\frac{b}{\delta_B}+a\right)(1-\delta) - \frac{b+a\delta_S\frac{1+\pi^*}{2}}{\delta_B}(1-\delta)}{1-\pi^*} + \frac{b+a\delta_S\frac{1+\pi^*}{2}}{\delta_B}(1-\delta) - a\delta_S\frac{1+\pi^*}{2}\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)$$
$$=\pi^* a\left((1-\delta)\frac{\delta_S}{\delta_B} - \left(1-\hat{\delta}+\hat{h}\delta_S\right)\right) + \frac{b+a\delta_S/2}{\delta_B}(1-\delta).$$

Simplify terms further

$$\frac{\left(\frac{b}{\delta_B}+a-\frac{b+a\delta_S\frac{1+\pi^*}{2}}{\delta_B}\right)(1-\delta)}{1-\pi^*} + \frac{a\delta_S\pi^*}{2\delta_B}(1-\delta) - a\delta_S\frac{1+\pi^*}{2}\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)$$
$$=\pi^* a\left((1-\delta)\frac{\delta_S}{\delta_B} - \left(1-\hat{\delta}+\hat{h}\delta_S\right)\right).$$

Simplify terms further

$$\frac{\delta_B-\delta_S(1+\pi^*)/2}{\delta_B(1-\pi^*)}(1-\delta) + \frac{\pi^*\delta_S}{2\delta_B}(1-\delta) - \frac{1+\pi^*}{2}\left(1-\hat{\delta}+\hat{h}\delta_S\right) = \pi^*\left((1-\delta)\frac{\delta_S}{\delta_B} - \left(1-\hat{\delta}+\hat{h}\delta_S\right)\right)$$
$$\frac{\delta_B-\delta_S(1+\pi^*)/2}{\delta_B(1-\pi^*)}(1-\delta) + \frac{\pi^*\delta_S}{2\delta_B}(1-\delta) + \frac{\pi^*-1}{2}\left(1-\hat{\delta}+\hat{h}\delta_S\right) = \pi^*(1-\delta)\frac{\delta_S}{\delta_B}$$
$$\frac{1-\frac{\delta_S}{\delta_B}\frac{1+\pi^*}{2}}{1-\pi^*} + \frac{\pi^*-1}{2}\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta}\right) = \pi^*\frac{\delta_S}{2\delta_B}$$
$$2-\frac{\delta_S}{\delta_B}(1+\pi^*) - (1-\pi^*)^2\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta)}\right) = \pi^*(1-\pi^*)\frac{\delta_S}{\delta_B}.$$

Moving terms to the right hand side obtain the expression in the Lemma. □

Proposition 3, part 5: optimality of the two-period test in the relaxed problem.

Lemma A.7 (No presale deviations)**.** *Given continuation function* $\bar{U}^*(\pi; b, a)$*, asset presales are suboptimal, i.e.,*

$$\bar{U}^*(\pi; b, a) = \max_{\hat{q}_S \in [0,a]} \bar{U}^*\left(\pi; \quad b + \hat{q}_S \cdot \delta_S \frac{1+\pi}{2}, \quad a - \hat{q}_S\right).$$

Proof. First, suppose that $\pi > \pi^* = \pi_{DF}$. Then the value function is given by

$$\begin{aligned}
\text{cav}_1\left[\bar{U}\right](\pi; b, a) &= \frac{\pi - \pi_{DF}}{1 - \pi_{DF}} \cdot \bar{u}_1(b, a) \\
&+ \frac{1-\pi}{1-\pi_{DF}} \cdot \left[\pi_{DF}\bar{u}_1\left(b + \delta_S \frac{1+\pi_{DF}}{2}, 0\right) + (1-\pi_{DF})\bar{u}_0\left(b + \delta_S \frac{1+\pi_{DF}}{2}, 0\right)\right] \\
&= \frac{1}{\delta_B} \cdot \left(b + a\delta_S \frac{1+\pi}{2} - (1-\pi)d + a(\delta_B - \delta_S) \cdot \frac{b + a\delta_S \frac{1+\pi}{2} - d}{b + a\delta_S - d}\right)
\end{aligned}$$

Presales change $a \to a - q$ and $b \to b + q\delta_S \frac{1+\pi}{2}$, and the relevant term in the expression above becomes

$$\frac{a}{b + a\delta_S - d} \quad \to \quad \frac{a-q}{b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S - d}.$$

Taking the derivative

$$\begin{aligned}
\frac{\partial}{\partial q}\left[\frac{a-q}{b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S - d}\right] &= \frac{-\left(b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S - d\right) - \delta_S \frac{\pi-1}{2}(a-q)\big)}{\left(b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S - d\right)^2} \\
&= -\frac{b + a\delta_S - d + \delta_S \frac{\pi-1}{2} a}{\left(b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S - d\right)^2} \\
&= -\frac{b + a\delta_S \frac{1+\pi}{2} - d}{\left(b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S - d\right)^2} < 0.
\end{aligned}$$

Hence, presales prior concavification are suboptimal for $\pi > \pi^* = \pi_{DF}$.

Now, suppose that $\pi > \pi^*$ but $\pi^* < \pi_{DF}$. Then the expected payoff to a presale of magnitude q is

$$\begin{aligned}
&\frac{1-\pi}{1-\pi^*} \cdot \pi^* \cdot \bar{u}_1\left(b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S \frac{1+\pi^*}{2}, 0\right) \\
+&\frac{1-\pi}{1-\pi^*} \cdot (1-\pi^*) \cdot \bar{u}_0\left(b + q\delta_S \frac{1+\pi}{2} + (a-q)\delta_S \frac{1+\pi^*}{2}, 0\right)
\end{aligned}$$

$$+\frac{\pi-\pi^*}{1-\pi^*}\cdot\bar{u}_1\left(b+q\delta_S\frac{1+\pi}{2},a-q\right)$$

Substituting the closed form expressions obtain (note that we're using the closed form subsequent presale being equal to $a-q$ for $\pi<\pi^*$)

$$\begin{aligned}
&\frac{1-\pi}{1-\pi^*}\cdot\pi^*\cdot\frac{b+q\delta_S\frac{1+\pi}{2}+(a-q)\delta_S\frac{1+\pi^*}{2}}{\delta_B}\cdot(1-\delta)\\
&+\frac{1-\pi}{1-\pi^*}\cdot(1-\pi^*)\cdot\left(b+q\delta_S\frac{1+\pi}{2}+(a-q)\delta_S\frac{1+\pi^*}{2}-d\right)\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)\\
&+\frac{\pi-\pi^*}{1-\pi^*}\cdot\left(\frac{b+q\delta_S\frac{1+\pi}{2}}{\delta_B}+a-q\right)\cdot(1-\delta)
\end{aligned}\tag{A.53}$$

Simplifying terms in (A.53), the marginal value of a presale is given by

$$\begin{aligned}
&\frac{1-\pi}{1-\pi^*}\pi^*\frac{\delta_S\frac{\pi-\pi^*}{2}}{\delta_B}(1-\delta)+(1-\pi)\delta_S\frac{\pi-\pi^*}{2}\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)+\frac{\pi-\pi^*}{1-\pi^*}\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)(1-\delta)\\
=&\frac{1-\pi}{1-\pi^*}\pi^*\frac{\delta_S}{2\delta_B}(1-\delta)+(1-\pi)\frac{\delta_S}{2}\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)+\frac{1}{1-\pi^*}\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)(1-\delta)\\
=&\frac{1-\pi}{1-\pi^*}\left(\pi^*\frac{\delta_S}{2\delta_B}(1-\delta)+(1-\pi^*)\left(\frac{1-\hat{\delta}}{2}+\frac{\hat{h}\delta_S}{2}\right)\right)+\frac{1}{1-\pi^*}\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)(1-\delta)\\
=&\frac{1-\pi}{1-\pi^*}\left(\pi^*\frac{\delta_S}{2\delta_B}(1-\delta)+(1-\pi^*)\left(\frac{1-\hat{\delta}}{2}+\frac{\hat{h}\delta_S}{2}\right)\right)+\frac{1}{1-\pi^*}\left(\frac{\delta_S}{\delta_B}\frac{1+\pi}{2}-1\right)(1-\delta)\\
=&\frac{1-\pi}{1-\pi^*}\left(\pi^*\frac{\delta_S}{\delta_B}+(1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta}\right)\right)+\frac{1}{1-\pi^*}\left(\frac{\delta_S}{\delta_B}(1+\pi)-2\right).
\end{aligned}\tag{A.54}$$

Multiplying (A.54) by $1-\pi^*$, obtain

$$\begin{aligned}
&(1-\pi^*+\pi^*-\pi)\left[\pi^*\frac{\delta_S}{\delta_B}+(1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta}\right)\right]+\frac{\delta_S}{\delta_B}(1+\pi)-2\\
=&(\pi^*-\pi)\left[\pi^*\frac{\delta_S}{\delta_B}+(1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta}\right)\right]\\
&+(1-\pi^*)\left[\pi^*\frac{\delta_S}{\delta_B}+(1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta}\right)\right]+\frac{\delta_S}{\delta_B}(1+\pi)-2\\
\overset{(i)}{=}&(\pi^*-\pi)\left[\pi^*\frac{\delta_S}{\delta_B}+(1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta}+\frac{\hat{h}\delta_S}{1-\delta}\right)\right]+2-\frac{\delta_S}{\delta_B}(1+\pi^*)+\frac{\delta_S}{\delta_B}(1+\pi)-2
\end{aligned}\tag{A.55}$$

$$
\begin{aligned}
=&(\pi^* - \pi)\left[\pi^*\frac{\delta_S}{\delta_B} + (1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta} + \frac{\hat{h}\delta_S}{1-\delta}\right)\right] + \frac{\delta_S}{\delta_B}(\pi - \pi^*) \\
=&(\pi - \pi^*)\left[-\pi^*\frac{\delta_S}{\delta_B} - (1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta} + \frac{\hat{h}\delta_S}{1-\delta}\right) + \frac{\delta_S}{\delta_B}\right] \\
=&(\pi - \pi^*)(1-\pi^*)\left[\frac{\delta_S}{\delta_B} - \left(\frac{1-\hat{\delta}}{1-\delta} + \frac{\hat{h}\delta_S}{1-\delta}\right)\right] \overset{(ii)}{<} 0, \qquad (A.56)
\end{aligned}
$$

where equality (i) in (A.55) holds by (A.52) in Lemma A.6 and inequality (ii) in (A.56) holds for any $\hat{h}$ since $\hat{\delta} > \delta$.

Case 3: Finally, for $\pi < \pi^*$ we have $\text{cav}[U] = U$, i.e. the value function is concave and there is no information revelation prior to adjusting bank's portfolio. Since $\pi^* \leq \min(\pi_{DF}, \bar{\pi})$ presale in this case is weakly suboptimal (since it will be followed by further complete sale $q(\pi) = a$ according to Lemma A.4. □

Lemma A.8 (No prepurchase deviations)**.** *Given continuation function $\bar{U}^*(\pi; b, a)$, asset presales are suboptimal, i.e.,*

$$
\bar{U}^*(\pi; b, a) = \max_{\hat{q}_B \in [0, 2b/\delta_B(1+\pi)]} \bar{U}^*\left(\pi; \quad b - \hat{q}_B \cdot \delta_B \frac{1+\pi}{2}, \quad a + \hat{q}_B\right).
$$

Proof. Case 1: $\pi > \pi^* = \pi_{DF}$.

$$
\begin{aligned}
\text{cav}_1\left[\bar{U}\right](\pi; b, a) &= \frac{1}{\delta_B} \cdot \left(b + a\delta_S\frac{1+\pi}{2} - (1-\pi)d + a(\delta_B - \delta_S) \cdot \frac{b + a\delta_S\frac{1+\pi}{2} - d}{b + a\delta_S - d}\right) \\
&= \frac{1}{\delta_B} \cdot \left(b + a\delta_B\frac{1+\pi}{2} - a(\delta_B - \delta_S)\frac{1+\pi}{2} - (1-\pi)d + a(\delta_B - \delta_S) \cdot \frac{b + a\delta_S\frac{1+\pi}{2} - d}{b + a\delta_S - d}\right) \\
&= \frac{1}{\delta_B} \cdot \left(b + a\delta_B\frac{1+\pi}{2} - (1-\pi)d + a(\delta_B - \delta_S) \cdot \left(-\frac{1+\pi}{2} + \frac{b + a\delta_S\frac{1+\pi}{2} - d}{b + a\delta_S - d}\right)\right) \\
&= \frac{1}{\delta_B} \cdot \left(b + a\delta_B\frac{1+\pi}{2} - (1-\pi)d + a(\delta_B - \delta_S) \cdot \left(\frac{1-\pi}{2} \cdot \frac{b-d}{b + a\delta_S - d}\right)\right) \\
&= \frac{1}{\delta_B} \cdot \left(b + a\delta_B\frac{1+\pi}{2} - (1-\pi)d - \frac{1-\pi}{2}(\delta_B - \delta_S) \cdot \frac{a(d-b)}{b + a\delta_S - d}\right)
\end{aligned}
$$

Notice that under pre-purchases only the term $\frac{a(d-b)}{b+a\delta_S-d}$ changes. The quantity of asset $a > 0$ goes up, the cash gap $d - b > 0$ goes up and $b + a\delta_S - d > 0$ goes down. Overall, the whole ratio increases with prepurchases, hence, the value function goes down (because of $-\frac{1-\pi}{2}(\delta_B - \delta_S)$ term). Hence prepurchases of the asset are also suboptimal for $\pi > \pi_{DF}$.

Case 2: $\pi > \pi^*$ but $\pi^* < \pi_{DF}$. The expected value from pre-purchasing q units of the asset is

$$
\begin{aligned}
&\frac{1-\pi}{1-\pi^*} \cdot \pi_* \cdot \frac{b - q\delta_B\frac{1+\pi}{2} + (a+q)\delta_S\frac{1+\pi^*}{2}}{\delta_B} \cdot (1-\delta) \\
+&\frac{1-\pi}{1-\pi^*} \cdot (1-\pi_*) \cdot \left(b - q\delta_B\frac{1+\pi}{2} + (a+q)\delta_S\frac{1+\pi^*}{2} - d\right)\left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h}\right) \\
+&\frac{\pi-\pi^*}{1-\pi^*} \cdot \left(\frac{b - q\delta_B\frac{1+\pi}{2}}{\delta_B} + a + q\right) \cdot (1-\delta)
\end{aligned}
$$

The marginal value of a pre-purchase is then given by

$$
\begin{aligned}
&\frac{1-\pi}{1-\pi^*} \cdot \pi_* \cdot \frac{-\delta_B\frac{1+\pi}{2} + \delta_S\frac{1+\pi^*}{2}}{\delta_B} \cdot (1-\delta) + \frac{1-\pi}{1-\pi^*} \cdot (1-\pi_*) \cdot \left(-\delta_B\frac{1+\pi}{2} + \delta_S\frac{1+\pi^*}{2}\right)\left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h}\right) \\
+&\frac{\pi-\pi^*}{1-\pi^*} \cdot \left(\frac{-\delta_B\frac{1+\pi}{2}}{\delta_B} + 1\right) \cdot (1-\delta) \\
=&\frac{1-\pi}{1-\pi^*}\left(-\delta_B\frac{1+\pi}{2} + \delta_S\frac{1+\pi^*}{2}\right)\left(\pi^*\frac{1-\delta}{\delta_B} + (1-\pi^*)\left(\frac{1-\hat{\delta}}{\delta_S} + \hat{h}\right)\right) + \frac{\pi-\pi^*}{1-\pi^*}\frac{1-\pi}{2}(1-\delta)
\end{aligned}
\tag{A.57}
$$

Dividing (A.57) by $\frac{1-\pi}{2}(1-\delta)$ obtain

$$
\begin{aligned}
&\frac{1}{1-\pi^*}\left(\delta_S(1+\pi^*) - \delta_B(1+\pi)\right)\left(\frac{\pi^*}{\delta_B} + (1-\pi^*)\left(\frac{1-\hat{\delta}}{\delta_S(1-\delta)} + \frac{\hat{h}}{1-\delta}\right)\right) + \frac{\pi-\pi^*}{1-\pi^*} \\
=&\frac{1}{1-\pi^*}\left(1+\pi^* - \frac{\delta_B}{\delta_S}(1+\pi)\right)\left(\pi^*\frac{\delta_S}{\delta_B} + (1-\pi^*)\left(\frac{1-\hat{\delta}}{1-\delta} + \frac{\hat{h}\delta_S}{1-\delta}\right)\right) + \frac{\pi-\pi^*}{1-\pi^*} \\
=&\frac{1}{(1-\pi^*)^2}\left(1+\pi^* - \frac{\delta_B}{\delta_S}(1+\pi)\right)\left(2 - \frac{\delta_S}{\delta_B}(1+\pi^*)\right) + \frac{\pi-\pi^*}{1-\pi^*} \\
\overset{(i)}{<}&\frac{1}{(1-\pi^*)^2}\left(1+\pi^* - 1\cdot(1+\pi)\right)\left(2 - \frac{\delta_S}{\delta_B}(1+\pi^*)\right) + \frac{\pi-\pi^*}{1-\pi^*} && \text{(A.58)} \\
=&\frac{\pi-\pi^*}{1-\pi^*}\left[-\frac{1}{1-\pi^*}\left(2 - \frac{\delta_S}{\delta_B}(1+\pi^*)\right) + 1\right] \\
=&\frac{\pi-\pi^*}{(1-\pi^*)^2}\left[-2 + \frac{\delta_S}{\delta_B}(1+\pi^*) + 1 - \pi^*\right] = \frac{(\pi-\pi^*)(1+\pi^*)}{(1-\pi^*)^2}\left(\frac{\delta_S}{\delta_B} - 1\right) \overset{(ii)}{<} 0, && \text{(A.59)}
\end{aligned}
$$

where inequalities (i) and (ii) in (A.58) and (A.59) both follow from $\delta_B > \delta_S$.

Case 3: $\pi < \pi^* \le \min(\pi_{DF}, \bar{\pi})$. In this region $\text{cav}\left[\bar{U}\right] = \bar{U}$, and for $\bar{U}$ it is optimal to sell everything without disclosing information according to Lemma A.4. Hence, prepurchase is strictly suboptimal. □

Lemma A.9 (Two period test optimality)**.** *Suppose $\pi > \pi^*$. The optimal sequential stress test in the relaxed problem can be implemented in two steps.*

Proof. Suppose $\mathcal{S} = \{S_n, R_n(\cdot)\}_{n=1}^N$ is an optimal sequential stress test in the relaxed problem. Denote by $(\pi_n)_{n=0}^N$ the belief process implemented by the stress test, where π_0 denotes the initial prior. It is without loss to assume that the optimal sequential test fully discloses θ eventually, i.e., $\pi_N \in \{0, 1\}$. If this were not the case, then the sequential test can be weakly improved upon by adding an additional step that reveals θ, followed by the adjustment in the bank's portfolio conditional on θ. Hence, it is without loss to assume that $S_N = \theta$.

Denote by (b_n, a_n) the portfolio held by the bank at the end of step n. In this notation, $(b_0, a_0) = (b, a)$ is the initial portfolio of the bank. The triple (π_n, b_n, a_n) summarizes the state of the bank at step n and are the only payoff relevant states. A backward induction argument, formalized below, ensures that the optimal stress test is Markov in the triplet (π, b, a). Formally, denote by $V_n(\pi_n, b_n, a_n)$ the expected value from an optimal stress test at the end of step n in state (π_n, b_n, a_n). Because $\pi_N \in \{0, 1\}$, the optimal capital requirements should implement the optimal portfolio conditional on θ, it implies that the continuation value at step $N-1$ is the expected value from disclosing θ fully and setting an optimal portfolio

$$V_{N-1}(\pi_{N-1};\ b_{N-1},\ a_{N-1}) = \pi_{N-1} \cdot \bar{u}_1(b_{N-1}, a_{N-1}) + (1 - \pi_{N-1}) \cdot \bar{u}_0(b_{N-1}, a_{N-1}). \tag{A.60}$$

The expected value (A.60) is the base of induction for the Markov structure of the problem. Suppose at the end of step $N-2$, the Markov state is $(\pi_{N-2}, b_{N-2}, a_{N-2})$. Then, given signal S_{N-1} and associated capital requirements $R_{N-1}(S_{N-1})$ the expected continuation value to the regulator is

$$V_{N-2}(\pi_{N-2}; b_{N-2}, a_{N-2}) = \mathrm{E}\left[V_{N-1}\Big(\mathrm{E}\left[\theta \middle| S_{N-1}\right]; b_{N-1}(S_{N-1}), a_{N-1}(S_{N-1})\Big) \;\Big|\; \pi_{N-2}, b_{N-2}, a_{N-2}\right]$$

$$\overset{(i)}{\leq} \mathrm{E}\Bigg[\underbrace{\max_{\hat{q}_S,\hat{q}_B}\left\{V_{N-1}\left(\mathrm{E}\left[\theta\middle|S_{N-1}\right];b_{N-2}+(\hat{q}_S\delta_S-\hat{q}_B\delta_B)\cdot\frac{1+\mathrm{E}\left[\theta|S_{N-1}\right]}{2},a_{N-2}-\hat{q}_S+\hat{q}_B\right)\right\}}_{=\bar{U}\left(\mathrm{E}[\theta|S_{N-1}];\,b_{N-2},\,a_{N-2}\right)}\ \Bigg|\ \pi_{N-2}\Bigg]$$

where the maximum is taken with respect to all feasible portfolios $(\hat{q}_S, \hat{q}_B)$ and inequality (i) holds because $(b_{N-1}(S_{N-1}), a_{N-1}(S_{N-1}))$ is a feasible portfolio. The optimality of the stress test requires that (i) must be binding P-a.s. Using the definition of $\bar{U}(\pi; b, a)$, the expected continuation value can be written as

$$V_{N-2}\Big(\mathrm{E}\left[\theta\middle|S_{N-2}\right];\ b_{N-2},\ a_{N-2}\Big) = \mathrm{E}\left[\bar{U}\left(\mathrm{E}\left[\theta\middle|S_{N-1}\right];\ b_{N-2},\ a_{N-2}\right)\ \middle|\ \pi_{N-2}\right]$$

$$\overset{(i)}{\leq} \max_{\tilde{\pi}\in\Delta(\pi_{N-2})}\left\{\mathrm{E}\left[\bar{U}\left(\tilde{\pi};\ b_{N-2},\ a_{N-2}\right)\right]\right\} \tag{A.61}$$

$$\overset{(ii)}{=} \mathrm{cav}_1\left[\bar{U}\left(\pi;\ b_{N-2},\ a_{N-2}\right)\right]\left(\pi_{N-2}\right) \tag{A.62}$$

$$\overset{(iii)}{=} \bar{U}^*(\pi_{N-2};\ b_{N-2},\ a_{N-2}). \tag{A.63}$$

where (i) in (A.61) holds because $\mathrm{E}\left[\theta|S_{N-1}\right] \in \Delta(\pi_{N-2})$, which denotes the set of distributions over $[0, 1]$ with mean π_{N-2}. Equality (ii) in (A.62) holds by the Bayesian persuasion arguments expressed in Kamenica and Gentzkow (2011), showing that the concavification operator delivers the expected value corresponding to the optimal signal chosen by the regulator. Finally, equality (iii) in (A.63) holds by definition of $\bar{U}^*(\pi; b, a)$. Thus, the value function under any optimal stress test satisfies

$$V_{N-2}\Big(\mathrm{E}\left[\theta\middle|S_{N-2}\right];\ b_{N-2},\ a_{N-2}\Big) \equiv \bar{U}^*(\pi_{N-2};\ b_{N-2},\ a_{N-2}).$$

Now, we show that for any $n < N-2$ it follows that

$$V_n\Big(\mathrm{E}\left[\theta\middle|S_n\right];\ b_n,\ a_n\Big) \equiv \bar{U}^*(\pi_n;\ b_n,\ a_n). \tag{A.64}$$

Equality (A.64) is shown for $n = N-2$. Suppose (A.64) holds for $n < N-2$. Then, by the previous argument, the optimal stress test satisfies

$$V_{n-1}(\pi_{n-1}; b_{n-1}, a_{n-1}) = \mathrm{E}\left[V_n\Big(\mathrm{E}\left[\theta\middle|S_n\right]; b_{n-1}(S_n), a_{n-1}(S_n)\Big)\ \middle|\ \pi_{n-1}, b_{n-1}, a_{n-1}\right]$$

$$\overset{(i)}{\leq} \mathrm{E}\left[\left[\max_{\hat{q}_S,\hat{q}_B}\left\{V_n\left(\mathrm{E}\left[\theta\middle|S_n\right];b_{n-1}+(\hat{q}_S\delta_S-\hat{q}_B\delta_B)\frac{1+\mathrm{E}\left[\theta|S_n\right]}{2},a_{n-1}-\hat{q}_S+\hat{q}_B\right);\ b_{N-2},\ a_{N-2})\right\}\ \middle|\ \pi_{n-1}\right]$$

$$=\mathrm{E}\left[\max_{\hat{q}_S,\hat{q}_B}\left\{\bar{U}^*\left(\mathrm{E}\left[\theta\middle|S_n\right];b_{n-1}+(\hat{q}_S\delta_S-\hat{q}_B\delta_B)\frac{1+\mathrm{E}\left[\theta|S_n\right]}{2},a_{n-1}-\hat{q}_S+\hat{q}_B\right);\ b_{N-2},\ a_{N-2})\right\}\ \middle|\ \pi_{n-1}\right]$$

$$=\mathrm{E}\left[\bar{U}^*(\mathrm{E}\left[\theta|S_n\right];b_{n-1},a_{n-1})\ \middle|\ \pi_{n-1}\right]\overset{(ii)}{\leq}\max_{\tilde{\pi}\in\Delta(\pi_{n-1})}\mathrm{E}\left[\bar{U}^*(\tilde{\pi};\ b_{n-1},\ a_{n-1})\right]$$

$$=\mathrm{cav}_1\left[\bar{U}^*(\pi;\ b_{n-1},\ a_{n-1}\right](\pi_{n-1})=\bar{U}^*(\pi_{n-1};\ b_{n-1},\ a_{n-1}).$$

Because $\bar{U}^*(\pi_{n-1};\ b_{n-1},\ a_{n-1})$ is an attainable payoff to a sequential stress test, the optimality of $\mathcal{S}$ implies that both inequalities (i) and (ii) must be binding, implying that $V_{n-1}(\pi_{n-1};b_{n-1},a_{n-1}) = \bar{U}^*(\pi_{n-1};b_{n-1},a_{n-1})$. Continuing the induction process, obtain that the ex-ante payoff of the optimal sequential stress test in the relaxed problem is given by $V_0(\pi_0;b_0,a_0) = \bar{U}^*(\pi_0;b_0,a_0)$. Q.E.D.

□

Proposition 3, part 6: attainability in the original problem.

Lemma A.10. *Suppose*

$$H'(0)\geq\frac{1-\delta}{\delta_S}\frac{1}{d-b}\left[\frac{d-b}{\delta_S}+\frac{a}{2}\frac{b/\delta_B+a}{b+a\delta_S-d}-\frac{b+a\delta_S/2}{\delta_B}\right].\tag{A.65}$$

Then $\pi^*=\pi_{DF}$.

Proof. From Lemma A.5 it follows that $\bar{U}^*(\pi;b,a)$ is concave for $\pi<\pi_{DF}$. The optimality of $\pi^*=\pi_{DF}$ is then given by a positive downward kink at $\pi=\pi_{DF}$ given by

$$\frac{\partial\bar{U}^*}{\partial\pi}(\pi-;b,a)\Big|_{\pi=\pi_{DF}}\leq\frac{U(1;b,a)}{1-\pi_{DF}}$$

$$\pi_{DF}a\underbrace{\left((1-\delta)\frac{\delta_S}{\delta_B}-\left(1-\hat{\delta}+\hat{h}\delta_S\right)\right)}_{<0}+(1-\delta)\cdot\frac{b+a\delta_S/2}{\delta_B}+(d-b)\left(\frac{1-\hat{\delta}}{\delta_S}+\hat{h}\right)\geq\frac{U(1;b,a)}{1-\pi_{DF}}$$

where the second inequality follows from the first via (A.51). A sufficient condition for the above to hold is

$$(1-\delta)\frac{b+a\delta_S/2}{\delta_B}+(d-b)\left(H'(0)\frac{\delta_S}{2}+H'(0)\sqrt{\frac{\delta_S^2}{4}-\frac{1-\delta}{H'(0)}}\right)\geq\frac{b/\delta_B+a}{1-\pi_{DF}}\cdot(1-\delta)$$

A sufficient condition for the above to hold (exploiting the concavity of the square root)

$$(1-\delta)\frac{b+a\delta_S/2}{\delta_B}+(d-b)\left(H'(0)\delta_S-\frac{1-\delta}{\delta_S}\right)\geq\frac{b/\delta_B+a}{1-\pi_{DF}}\cdot(1-\delta)$$

$$H'(0)\cdot\delta_S(d-b)\geq(1-\delta)\cdot\left[\frac{d-b}{\delta_S}+\frac{b/\delta_B+a}{2-\frac{2}{\delta_S}\frac{d-b}{a}}-\frac{b+a\delta_S/2}{\delta_B}\right]$$

$$H'(0)\geq\frac{1-\delta}{\delta_S}\frac{1}{d-b}\left[\frac{d-b}{\delta_S}+\frac{a}{2}\frac{b/\delta_B+a}{b+a\delta_S-d}-\frac{b+a\delta_S/2}{\delta_B}\right]$$

□

Corollary A.2. *Suppose* $\delta_S\frac{1+\pi}{2}\geq\frac{d-b}{a}\leq\frac{\delta_S}{2}$, (A.30) *and* (A.65) *are satisfied. Then, the expected value to the relaxed problem* $\bar{U}^*(\pi;b,a)$ *is attained via a two-step stress test.*

Proof. The optimal stress test in the relaxed problem implements a default-free allocation for the banks. The expected value is

$$\bar{U}^*(\pi;b,a)=\frac{\pi-\pi_{DF}}{1-\pi_{DF}}\cdot\left(\frac{b}{\delta_B}+a\right)\cdot(1-\delta)+\frac{\pi-\pi_{DF}}{1-\pi_{DF}}\cdot\pi_{DF}\cdot\frac{d}{\delta_B}\cdot(1-\delta).$$

This is exactly the allocation that is achieved by a stress test that first provides a signal with posteriors in $\{\pi_{DF},1\}$, and then engages in precautionary recapitalization. □

Corollary A.3. *If* $H'(0)$ *is linear, then a* $\pi^*<\pi_{DF}$ *relaxed stress test can be attained in the original model.*

A.8 Proof of Proposition 4

Consistent with the main text, we refer to $X_j=X$ as the low quality asset, and $X_j\equiv 1$ as the high quality asset. Also consistent with the main text, we refer to the bank which starts with asset $X_j=X$ as the weak bank, and the bank which starts with asset $X_j\equiv 1$ as the strong bank.

Proposition 4, part 1: optimal static stress test.

Suppose the market value of the bank's portfolio exceeds liabilities, i.e., $b + a \cdot \delta_S \frac{1+\mu_0}{2} \geq d$. The regulator can reduce the riskiness of the weak bank with $X_j = X$ in one of two ways - either by requiring a sale of some of its low quality asset, or by making it feasible for the weak bank to purchase some high quality asset at a discount from a strong bank. Lemma A.11 shows that failing safe banks to force them to sell their high quality assets to weak banks is sub-optimal.

Lemma A.11 (Static test under idiosyncratic risk)**.** *Suppose $b < d \cdot \delta_S$. The optimal stress test can be implemented with an adverse pass-fail test. The posterior belief about asset quality of the failing banks is given by*

$$\mathrm{P}(X_j = X \,|\, S = \mathit{fail}) = \pi_{DF} \stackrel{def}{=} \frac{2}{\delta_S} \cdot \frac{d-b}{a} - 1.$$

The number of strong banks being revealed is given by $\alpha \stackrel{def}{=} \frac{\mu - \pi_{DF}}{1 - \pi_{DF}}$.

Proof. For any stress test, consider the set of banks that pass the stress test, i.e., those that according to the prescribed capital ratios can acquire more risky assets (either from the failing banks or from the market). If the stress test passes a weak bank then this bank can, at most, purchase b/δ_S units of high quality assets from a strong bank that failed the stress test. Because the purchase is performed at a discount, it makes the weak bank safer, but makes it retain all of its initial low quality asset. Because the good quality asset pays 1 at time $t = 2$, while the worst outcome for the low quality asset is 0, the worst case payoff from such a portfolio is $b/\delta_S - d$. If $b/\delta_S < d$, then a weak bank cannot be safe if it passes the test. This rules out the possibility that weak banks be allowed to purchase assets after the stress test. Hence, only strong banks are able to buy the asset after a stress test, while all weak banks, and, perhaps, some strong banks, are required to sell some of their assets. The quantity and composition of such sales may depend on the exact failing grade that they get.

Next we show that there can only one failing grade. In order for the weak bank to avoid distress, the market value of a weak bank's assets must exceed liabilities. In other words, the posterior likelihood that the bank has low quality assets needs to be sufficiently high

$$\mathrm{P}(X_j = 1 \,|\, S_j) \overset{P\text{-}a.s.}{\geq} \pi_{DF}. \tag{A.66}$$

If this were not the case, then the weak banks end up in distress with positive probability to cover liabilities d. Suppose that (A.66) is strict with a positive probability. Consider then a modified stress test that splits this pool into two: "failing" pool with the fraction of strong banks π_{DF} and the remaining "passing" pool of strong banks. Consider an alternative signal $\hat{S}$ such that

$$\hat{S}_j = \begin{cases} \textit{pass} & \textit{if} \quad \mathrm{P}(X_j = 1 \,|\, S_j) = 1, \\ \textit{fail} & \textit{if} \quad \mathrm{P}(X_j = 1 \,|\, S_j) = \pi_{DF}, \\ \textit{fail} & \textit{if} \quad \mathrm{P}(X_j = 1 \,|\, S_j) > \pi_{DF} \textit{ and } X_j = X, \\ \textit{pass} \quad \textit{with probability} \quad \dfrac{\mathrm{P}(X_j = 1 \,|\, S_j) - \pi_{DF}}{\mathrm{P}(X_j = 1 \,|\, S_j) \cdot (1 - \pi_{DF})} & \textit{if} \quad \mathrm{P}(X_j = 1 \,|\, S_j) > \pi_{DF} \textit{ and } X_j = 1, \\ \textit{fail} \quad \textit{with probability} \quad \dfrac{\pi_{DF} \cdot (1 - \mathrm{P}(X_j = 1 \,|\, S_j))}{\mathrm{P}(X_j = 1 \,|\, S_j) \cdot (1 - \pi_{DF})} & \textit{if} \quad \mathrm{P}(X_j = 1 \,|\, S_j) > \pi_{DF} \textit{ and } X_j = 1. \end{cases}$$

Whenever $\mathrm{P}(X_j = 1 \,|\, S_j) < 1$, bank j needs to raise $d - b$ units of safe assets, but the size of the failing pool is lower under signal $\hat{S}$, as can be seen from

$$\begin{aligned} \mathrm{P}\left(\hat{S}_j = \textit{fail}\right) &= \mathrm{E}\left[\mathbb{1}\left\{\mathrm{P}(X_j = 1 \,|\, S_j) = \pi_{DF}\right\}\right] \\ &\quad + \mathrm{E}\Bigg[\underbrace{\frac{1 - \mathrm{P}(X_j = 1 \,|\, S_j)}{1 - \pi_{DF}}}_{<1} \cdot \mathbb{1}\left\{\mathrm{P}(X_j = 1 \,|\, S_j) > \pi_{DF}\right\}\Bigg] < \mathrm{E}\left[\mathbb{1}\left\{\mathrm{P}(X_j = 1 \,|\, S_j) < 1\right\}\right]. \end{aligned}$$

This implies that the expected amount of capital that needs to be raised by the failing banks is weaker under signal $\hat{S}$, once efficient capital requirements are imposed. Moreover, the increased disclosure of strong banks by signal $\hat{S}_j$ increases the number of strong banks which are allowed to purchase assets from the banks which fail the stress test, further improving the risk-allocation across banks. As the asset supply of the banks that fail the stress test decreases, and asset demand of the banks that pass the stress test increases, the efficiency of asset holdings in the economy increases as more of them are retained by banks and fewer sold to outside investors, who are second-best holders of the risky asset. □

In what follows we characterize the expected welfare of the optimal static stress test. The price of the failing banks' assets is $\delta_S \cdot \mathrm{E}\left[X_j | S_j = fail\right] = \frac{d-b}{a}$ allows the banks that fail the stress test to raise $d - b$ units of cash by selling all of their risky assets to either the banks that pass the stress test or to the market. If a passing bank buys a diversified pool of the assets from the failing banks then only π_{DF} of those pay 1 while $1 - \pi_{DF}$ of those might pay $X = 0$. Denote by q_S to be the quantity of such assets that the strong passing bank can purchase from the banks that fail the stress test without violating its own solvency constraint

$$b + a + q_S \cdot \left(\pi_{DF} - \frac{d-b}{a}\right) = d \qquad \Leftrightarrow \qquad q_S \overset{def}{=} \frac{b+a-d}{(d-b)/a - \pi_{DF}}. \tag{A.67}$$

Note that q_S is not taking into account the budget constraint of the strong bank that passes the stress test at this point. Ignoring this budget constraint, the strong banks are able to purchase all of the assets of the failing banks without violating their solvency constraint if and only if

$$\begin{aligned} \alpha \cdot q_S &> (1-\alpha) \cdot a \\ (\mu - \pi_{DF}) \cdot \frac{b+a-d}{(d-b)/a - \pi_{DF}} &\geq (1-\mu) \cdot a \\ (\mu - \pi_{DF}) \cdot (b+a-d) &\geq (1-\mu)(d-b-a\cdot\pi_{DF}) \\ (\mu - \pi_{DF} + 1 - \mu)(b-d) + a \cdot (\mu - \pi_{DF} + (1-\mu)\pi_{DF}) \geq 0 & \\ (1-\pi_{DF})(b-d) + a\cdot(\mu - \pi_{DF} + \pi_{DF} - \mu\pi_{DF}) \geq 0 & \\ b + a \cdot \mu \geq d, & \end{aligned} \tag{A.68}$$

where (A.68) holds by the parametric assumption in the formulation of Proposition 4. Parametric assumption $b + a \cdot \mu \geq d$ ensures that the strong passing banks' ability to purchase the assets of the banks which fail the stress test is not limited by their own solvency considerations under the optimal stress test.[49] This implies that the only restriction on the strong passing banks' ability to

[49]Note that it may be the case that the strong bank purchases bad quality asset from the capital market on top of what it purchases from the failing banks.

purchase assets from the failing banks is driven by their liquidity constraint.

The demand for the risky asset by the strong passing banks may also be limited by these banks' available liquidity. Every passing strong bank has b units of safe assets that can be used to purchase the risky assets of the banks that fail the stress test. Hence, the total available liquidity of the passing banks is $\alpha \cdot b$. The banks that fail the optimal stress test need to raise $d - b$ so that the weak banks avoid distress. Hence, the total demand for capital from the banks that fail the stress test is $(1-\alpha)(d-b)$ of capital.

Two cases are possible depending on whether the strong banks have sufficient liquidity or not.

- Case 1: strong banks that pass the stress test do not have sufficient liquidity to purchase all the risky asset from the banks that fail the stress test

$$\alpha \cdot b < (1-\alpha) \cdot (d-b) \qquad \Leftrightarrow \qquad b < \frac{1-\mu}{1-\pi_{DF}} \cdot d.$$

  In this case some assets must be sold to the outside investors. Expected welfare is

$$\begin{aligned} & a \cdot \mathrm{E}[X](1-\delta) + \frac{1-\delta}{\delta_S} \cdot \left[\alpha \cdot b - (1-\alpha) \cdot (d-b)\right] \\ =& a \cdot \frac{1+\mu}{2} \cdot (1-\delta) + \frac{1-\delta}{\delta_S} \cdot \left[b - \frac{1-\mu}{1-\pi_{DF}} \cdot d\right]. \end{aligned}$$

  In this case, the strong banks that pass the stress test do not acquire any additional low quality asset from the outside capital markets as all of their liquidity is used to purchase the assets of the banks that fail the stress test which are relatively cheaper (discount δ_S) and of better expected quality, than the risky assets held by outside investors initially.

- Case 2: strong banks that pass the stress test have sufficient liquidity to purchase all of the risky asset from the banks that fail the stress test

$$\alpha \cdot b \geq (1-\alpha) \cdot (d-b) \qquad \Leftrightarrow \qquad b \geq \frac{1-\mu}{1-\pi_{DF}} \cdot d.$$

  The strong banks that pass the stress test are able to buy all the assets of the failing banks and purchase more risky assets from the market. After purchasing all the risky asset from the banks that fail the stress test, each strong bank that passes the stress test has $a/\alpha = a \cdot \frac{1-\pi_{DF}}{\mu-\pi_{DF}}$ risky assets, a fraction μ of them being high quality. The residual cash available to each passing bank is

$$b - \frac{1-\alpha}{\alpha} \cdot (d-b) = \frac{b}{\alpha} - \frac{1-\alpha}{\alpha} \cdot d \geq 0.$$

The strong bank that passes the stress test can purchase low quality asset $\hat{q}_B$ from the outside market as long as it does not violate its solvency and budget constraints

$$\begin{cases} \frac{a}{\alpha} \cdot \mu + \frac{b}{\alpha} - \frac{1-\alpha}{\alpha} \cdot d - \hat{q}_B \cdot \frac{\delta_B}{2} \geq d, & \textit{(solvency constraint)} \\ \qquad\qquad \frac{b}{\alpha} - \frac{1-\alpha}{\alpha} \cdot d - \hat{q}_B \cdot \frac{\delta_B}{2} \geq 0. & \textit{(liquidity constraint)} \end{cases} \tag{A.69}$$

Multiplying constraints (A.69) by α and simplifying terms obtain

$$\begin{cases} a \cdot \mu + b - d \geq \hat{q}_B \cdot \alpha \cdot \frac{\delta_B}{2}, & \textit{(solvency constraint)} \\ b - d + \alpha \cdot d \geq \hat{q}_B \cdot \alpha \cdot \frac{\delta_B}{2}. & \textit{(liquidity constraint)} \end{cases}$$

This implies that each strong bank that pass the stress test can, at most, purchase

$$\hat{q}_B = \frac{2}{\alpha \delta_B} \cdot \min\Big\{ \alpha \cdot \mu + b - d, \qquad b - d + \alpha \cdot d \Big\}.$$

The expected social welfare of a single strong bank that passes the stress test is given by

$$\begin{aligned} &(1-\delta) \cdot \left[\frac{a}{\alpha} \cdot \frac{1+\mu}{2} + \frac{1}{\delta_B} \cdot \frac{1}{\alpha} \cdot \min\Big\{ a \cdot \mu + b - d, \quad b - d + \alpha d \Big\} \right] \\ =&(1-\delta) \cdot \left[\frac{a}{\alpha} \cdot \frac{1+\mu}{2} + \frac{1}{\delta_B} \cdot \frac{b-d}{\alpha} + \frac{1}{\delta_B} \cdot \frac{1}{\alpha} \cdot \min\Big\{ a \cdot \mu, \quad \alpha \cdot d \Big\} \right] \\ =&(1-\delta) \cdot \frac{1}{\alpha} \cdot \left[a \cdot \frac{1+\mu}{2} + \frac{b-d}{\delta_B} + \frac{1}{\delta_B} \cdot \min\Big\{ a \cdot \mu, \quad \alpha \cdot d \Big\} \right]. \end{aligned}$$

where $\alpha = \frac{\mu-\pi_{DF}}{1-\pi_{DF}}$ and $\pi_{DF} = \frac{2}{\delta_S} \cdot \frac{d-b}{a} - 1$. As there is a total of α banks that pass the stress test, the expected social welfare is equal to

$$(1-\delta)\cdot\left[a\cdot\frac{1+\mu}{2}+\frac{b-d}{\delta_B}+\frac{1}{\delta_B}\cdot\min\Big\{a\cdot\mu,\quad \alpha\cdot d\Big\}\right].$$

Proposition 4, part 2: conditions for optimality of precautionary recapitalization.

Lemma A.12 (Precautionary recapitalization under idiosyncratic risk)**.** *Precautionary recapitalization is optimal whenever* $b<(1-\alpha)\cdot d$.

Proof. Suppose the strong banks that pass the stress test do not have sufficient liquidity to purchase the risky assets of the banks that fail the stress test, i.e.,

$$\alpha\cdot b<(1-\alpha)\cdot(d-b)\qquad\Leftrightarrow\qquad b<\frac{1-\mu}{1-\pi_{DF}}\cdot d.$$

Consider a precautionary recapitalization of quantity $q>0$. It changes the banks' portfolios to $b+q\cdot\delta_S\mathrm{E}\left[X\right]$ of safe assets and $a-q$ of risky assets. Denote by $\mu_{DF}(q)$ the default-free threshold given presale q as

$$\begin{aligned}\delta_S\cdot\frac{1+\mu_{DF}(q)}{2}&=\frac{d-b-q\cdot\delta_S(1+\mu)/2}{a-q}\\ \mu_{DF}(q)&\overset{def}{=}\frac{2}{\delta_S}\cdot\frac{d-b-q\cdot\delta_S(1+\mu)/2}{a-q}-1.\end{aligned}$$

Denote by $\alpha(q)$ to be the number of strong banks the stress test can pass following such precautionary recapitalization. Then

$$\begin{aligned}\alpha(q)&\overset{def}{=}\frac{\mu-\mu_{DF}(q)}{1-\mu_{DF}(q)}=\frac{1+\mu-\frac{2}{\delta_S}\cdot\frac{d-b-q\cdot\delta_S(1+\mu)/2}{a-q}}{2-\frac{2}{\delta_S}\cdot\frac{d-b-q\cdot\delta_S(1+\mu)/2}{a-q}}=\frac{\delta_S\frac{1+\mu}{2}-\frac{d-b-q\cdot\delta_S(1+\mu)/2}{a-q}}{\delta_S-\frac{d-b-q\cdot\delta_S(1+\mu)/2}{a-q}}\\ &=\frac{b+a\cdot\delta_S(1+\mu)/2-d}{b+a\delta_S-d+q\cdot\delta_S(-1+(1+\mu)/2)}=\frac{b+a\cdot\delta_S(1+\mu)/2-d}{b+a\delta_S-d-q\cdot\delta_S(1-\mu)/2}.\end{aligned}\tag{A.70}$$

A higher pre-sale q increase the number of strong banks that pass the test $\alpha(q)$. For subsequent derivations it is convenient to explicitly write out

$$
\begin{aligned}
1-\alpha(q) &= 1-\frac{b+a\cdot\delta_S(1+\mu)/2-d}{b+a\delta_S-d-q\cdot\delta_S(1-\mu)/2} \\
&= \frac{b+a\delta_S-d-q\cdot\delta_S(1-\mu)/2-b-a\cdot\delta_S(1+\mu)/2+d}{b+a\delta_S-d-q\cdot\delta_S(1-\mu)/2} \\
&= \frac{(a-q)\cdot\delta_S(1-\mu)/2}{b+a\delta_S-d-q\cdot\delta_S(1-\mu)/2}.
\end{aligned}
$$

A q presale implies that there subsequently exists $q\cdot\mu$ units of the high quality asset held by outside investors that the banks could, potentially, reacquire albeit at a high discount. Suppose the magnitude of the presale is such that the banks who pass the stress test do not have enough liquidity to acquire all of the risky assets from the banks that fail the stress test

$$
b+q\cdot\delta_S\frac{1+\mu}{2}<(1-\alpha(q))\cdot d.
$$

The welfare of the optimal stress test following a q precautionary recapitalization is then given by

$$
\begin{aligned}
&(a-q)\cdot\mathrm{E}[X]+q\cdot\delta\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[\alpha(q)\cdot\Big(b+q\cdot\delta_S\mathrm{E}[X]\Big)-(1-\alpha(q))\cdot\Big(d-b-q\cdot\delta_S\mathrm{E}[X]\Big)\Big] \\
=&(a-q)\cdot\mathrm{E}[X]+q\cdot\delta\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[b+q\cdot\delta_S\mathrm{E}[X]-(1-\alpha(q))\cdot d\Big] \\
=&a\cdot\mathrm{E}[X]-q\cdot(1-\delta)\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[b+q\cdot\delta_S\mathrm{E}[X]-(1-\alpha(q))\cdot d\Big] \\
=&a\cdot\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[b-(1-\alpha(q))\cdot d\Big]=a\cdot\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[b-d\Big]+\alpha(q)\cdot\frac{1-\delta}{\delta_S}\cdot d \\
\overset{(i)}{>}&a\cdot\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[b-d\Big]+\alpha(0)\cdot\frac{1-\delta}{\delta_S}\cdot d=a\cdot\mathrm{E}[X]+\frac{1-\delta}{\delta_S}\cdot\Big[b-(1-\alpha(0))\cdot d\Big].
\end{aligned}
$$

where inequality (i) follows from $\alpha(q)$ increasing in q as can be seen from (A.70). The welfare is increasing in $\alpha(q)$ which, in turn, is increasing in q. Precautionary recapitalization is, thus, profitable as it allows to disclose more strong banks who, in turn, can purchase more of the assets from the weak banks. This implies that the optimal presale quantity q must be at least large enough so that

$$
b+q\cdot\delta_S\frac{1+\mu}{2}\geq(1-\alpha(q))\cdot d.
$$

□

Lemma A.13 (Interaction between Interbank Trade and Precautionary Recapitalization)**.** *Suppose there is only idiosyncratic risk among banks with fraction μ of them holding high quality risky asset.*

- *In the presence of interbank trade, precautionary recapitalization is beneficial under the default-free test for all $\mu < \mu_{liq}$, where threshold μ_{liq} solves*

$$b + a \cdot \delta_S \frac{1+\mu_{liq}}{2} - d \stackrel{def}{=} (b + a \cdot \delta_S - d) \cdot \frac{d-b}{d}.$$

- *In the absence of interbank trade, precautionary recapitalization is beneficial under the default-free test whenever $\mu < \bar{\mu}$, where $\bar{\mu}$ solves*

$$b + a \cdot \delta_S \frac{1+\bar{\mu}}{2} - d \stackrel{def}{=} \frac{\delta_S}{\delta_B - \delta_S} \cdot \frac{1-\bar{\mu}}{2} \cdot d.$$

Moreover, $\mu_{liq} > \bar{\mu}$ for low b and $\mu_{liq} < \bar{\mu}$ for high b.

Proof. Default-free test passes a fraction $\alpha = \frac{\mu-\mu_{DF}}{1-\mu_{DF}}$ of banks (all of which are strong) and fails $1-\alpha$ banks that have to sell all their risky assets. The distinction between the two cases arises from whether the banks that pass the test have the option to purchase assets from banks that fail the test.

First, suppose interbank trade present. When $b < (1-\alpha) \cdot d$ the passing strong banks do not have enough liquidity to buy all the risky asset from the banks that fail the stress test and the precautionary recapitalization is welfare improving. The number of failing banks is sufficiently high for the strong passing banks to not be able to acquire all of their assets if

$$b < d(1-\alpha)$$

$$\alpha d < d - b$$

$$\frac{\mu - \mu_{DF}}{1-\mu_{DF}} \cdot d < d - b$$

$$\frac{b - a\delta_S \frac{1+\mu}{2} - d}{b + a\delta_S - d} < \frac{d-b}{d}$$

$$b - a\delta_S \frac{1+\mu}{2} - d < (b + a\delta_S - d)\frac{d-b}{d} \tag{A.71}$$

Inequality (A.71) proves the first part of the lemma and pins down μ_{liq} such that for $\mu < \mu_{liq}$ precautionary recapitalization is welfare improving.

Suppose now that the interbank market is not present now. Then expected welfare in this case is simply given by

$$\alpha(1-\delta)\left(a + \frac{b}{\delta_B}\right). \tag{A.72}$$

as the banks that pass the stress test can only purchase the asset from the capital market. In this expression of welfare we have assumes that the passing strong banks' balance sheet constraint is not binding, i.e., they can spend all their liquidity b on purchasing risky asset from the market (regardless of its quality). This way, it does not matter whether these banks are purchasing high or low quality asset from the market, just that they are purchasing it at the discount δ_B. The expected welfare (A.72) can be rewritten as

$$\alpha(1-\delta)\left(a + \frac{b}{\delta_B}\right) = \frac{b + a\delta_S\frac{1+\mu}{2} - d}{b + a\delta_S - d} \cdot \frac{1-\delta}{\delta_B}(b + a\delta_B) = \underbrace{\frac{1-\delta}{\delta_B}\left(b + a\delta_S\frac{1+\mu}{2} - d\right)}_{\textit{term } (i)} \cdot \underbrace{\frac{b + a\delta_B}{b + a\delta_S - d}}_{\textit{term } (ii)}.$$

The first term (i) is not affected by a marginal precautionary recapitalization as the latter does not change the market value of the bank's portfolio net of liabilities. The second term (ii), however, depends on the composition of the bank's portfolio. When the bank sells q units of the asset in a precautionary manner, the second term (ii) becomes

$$\frac{b + q\delta_S\frac{1+\mu}{2} + a\delta_B - q\delta_B}{b + q\delta_S\frac{1+\mu}{2} + a\delta_S - q\delta_S - d}.$$

To trace the net effect of a small precautionary recapitalization we only need to look at whether the above expression is increasing or decreasing in q around $q = 0$

$$\left(\frac{b + q\delta_S\frac{1+\mu}{2} + a\delta_B - q\delta_B}{b + q\delta_S\frac{1+\mu}{2} + a\delta_S - q\delta_S - d}\right)'_{q=0} = \frac{\left(\delta_S\frac{1+\mu}{2} - \delta_B\right)(b + a\delta_S - d) - \left(\delta_S\frac{1+\mu}{2} - \delta_S\right)(b + a\delta_B)}{(\ldots)^2}$$

$$
\begin{aligned}
&\sim \left(\delta_S \frac{1+\mu}{2} - \delta_S + \delta_S - \delta_B\right)(b + a\delta_S - d) - \left(\delta_S \frac{1+\mu}{2} - \delta_S\right)(b + a\delta_B) \\
&= \delta_S \frac{\mu - 1}{2}(a\delta_S - a\delta_B - d) + (\delta_S - \delta_B)(b + a\delta_S - d) \\
&= \delta_S \frac{1-\mu}{2} d + (\delta_S - \delta_B)\left(\delta_S \frac{\mu - 1}{2} a + b + a\delta_S - d\right) \\
&= \delta_S \frac{1-\mu}{2} d - (\delta_B - \delta_S)\left(b + a\delta_S \frac{1+\mu}{2} - d\right).
\end{aligned}
$$

Hence a small marginal recapitalization in the absence of interbank trade is welfare improving if

$$
b + a\delta_S \frac{1+\mu}{2} - d < \frac{\delta_S}{\delta_B - \delta_S} \cdot \frac{1-\mu}{2} \cdot d. \tag{A.73}
$$

Inequality (A.73) proves the second part of the lemma and pins down $\bar{\mu}$ when (A.73) is binding. Finally, note that for $b \approx 0$ $\mu_{liq} \approx 1$ while $\bar{\mu} < 1$. And for $b \approx d$ we have $\mu_{liq} < 0$ and $\bar{\mu} > 0$. This concludes the proof. □

Proposition 4, part 3: optimal sequential stress test.

Lemma A.14 (Sequential test under idiosyncratic risk)**.** *The optimal sequential stress test implements the expected welfare of the sequential stress test for the representative bank, given by*

$$
\textit{Representative Bank Welfare} = \begin{cases} \dfrac{1-\delta}{\delta_S} \cdot \left[a \cdot \left(\dfrac{\delta_S}{2} + \mu_0\left(1 + \dfrac{\delta_S}{2}\right)\right) + b - d\right] & \textit{if} \quad b + a\mu_0 \geq d, \\ \dfrac{1-\delta}{\delta_B} \cdot \left[a \cdot \left(\dfrac{\delta_B}{2} + \mu_0\left(1 + \dfrac{\delta_B}{2}\right)\right) + b - d\right] & \textit{if} \quad b + a\mu_0 < d. \end{cases}
$$

Proof. Consider a feasible ex-post allocation at the end of an N-step sequential stress test. Formally, denote $B_{i,N}$ as the quantity of the safe asset of bank i, and $A_{i,j,N}$ is the quantity of asset X_j held by bank i at the end of step N. Moreover, denote by $\hat{A}_{i,N}$ as the amount of bad asset (purchased from the outside market) held by bank i at the final step of the test.

At the final step N the default-free constraint for bank $i \in [0, 1]$ cab be expressed as

$$
B_{i,N} + \int_0^1 A_{i,j,N} \cdot \mathbb{1}\{X_j = 1\}\, dj \geq d.
$$

The expected welfare across all banks is given by

$$(1-\delta)\cdot \mathrm{E}\left[\int_0^1 X_j \int_0^1 A_{i,j,N}\, di\, dj + \frac{1}{2}\int_0^1 \hat{A}_{i,N}\, di\right].$$

Denote by $(B, A^0, A^1, \hat{A})$ the portfolio of the representative bank at the end of the sequential test, given by

$$\begin{cases} B \overset{def}{=} \int_0^1 B_{i,N}\, di, & \textit{(banks' cash holdings)} \\ A^0 \overset{def}{=} \int_0^1 \int_0^1 A_{i,j,N}\cdot \mathbb{1}\{X_j = X\}\, di\, dj, & \textit{(banks' low quality asset)} \\ A^1 \overset{def}{=} \int_0^1 \int_0^1 A_{i,j,N}\cdot \mathbb{1}\{X_j = 1\}\, di\, dj, & \textit{(banks' high quality assets)} \\ \hat{A} \overset{def}{=} \int_0^1 \hat{A}_{i,N}\, di. & \textit{(banks' low quality assets bought from market)} \end{cases} \tag{A.74}$$

The regulator's expected welfare from the sequential stress test can be expressed as

$$\mathrm{E}\left[A^1\cdot(1-\delta) + A^0\cdot \frac{1-\delta}{\delta_S} + \hat{A}\cdot \frac{1-\delta}{2}\right].$$

The representative bank is default-free as can be seen from

$$\begin{aligned} B + A^1 &= \int_0^1 B_{i,N}\, di + \int_0^1\int_0^1 A_{i,j,N}\cdot \mathbb{1}\{X_j = 1\}\, di\, dj \\ &= \int_0^1 \left(B_{i,N} + \int_0^1 A_{i,j,N}\, dj\right) di \geq \int_0^1 d\, dj = d. \end{aligned}$$

Moreover, any individual bank allocation maps to the aggregate portfolio $(B, A^0, A^1, \hat{A})$ of the representative bank. This implies that the optimal sequential stress test for the representative bank weakly dominates the expected payoff from a sequential test of the cross-section of banks involves a sale (or a purchase) at fundamental price $\delta_S/2$ (or $\delta_B/2$). The starting portfolio of the representative bank contains $a\cdot\mu_0$ units of high quality asset $X_j = 1$. Selling these assets to the outside market is strictly suboptimal - any stress test that involves a sale of this asset to the outside market can be improved. Consider two cases depending on whether the representative bank is buying or selling low quality assets from the capital market.

- Suppose that $b + a \cdot \mu_0 \leq d$. Then, the sale quantity q_S satisfies

$$\begin{aligned}
b + a \cdot \mu_0 + q_S \cdot \frac{\delta_S}{2} &= d, \\
q_S &= \frac{d - b - a \cdot \mu_0}{\delta_S/2}, \\
a(1-\mu_0) - q_S &= \frac{b + a \cdot (1-\mu_0)\delta_S/2 + a \cdot \mu_0 - d}{\delta_S/2}.
\end{aligned}$$

The expected welfare is then given by

$$\begin{aligned}
a \cdot \mu_0(1-\delta) + (a(1-\mu_0) - q_S) \cdot \frac{1-\delta}{2} &= a \cdot \mu_0(1-\delta) + \frac{b + a \cdot (1-\mu_0)\delta_S/2 + a \cdot \mu_0 - d}{\delta_S/2} \cdot \frac{1-\delta}{2} \\
&= \frac{1-\delta}{\delta_S} \cdot \left(a \cdot \mu_0 \delta_S + b + a \cdot (1-\mu_0)\frac{\delta_S}{2} + a \cdot \mu_0 - d \right) \\
&= \frac{1-\delta}{\delta_S} \cdot \left[a \cdot \left(\mu_0(1+\delta_S) + (1-\mu_0)\frac{\delta_S}{2} \right) + b - d \right] \\
&= \frac{1-\delta}{\delta_S} \cdot \left[a \cdot \left(\frac{\delta_S}{2} + \mu_0 \left(1 + \frac{\delta_S}{2} \right) \right) + b - d \right].
\end{aligned}$$

- Suppose that $b + a \cdot \mu_0 \leq d$. Then, the purchase quantity q_B satisfies

$$\begin{aligned}
b + a \cdot \mu_0 - q_B \cdot \frac{\delta_B}{2} &= d, \\
q_B &= \frac{b + a \cdot \mu_0 - d}{\delta_B/2}, \\
a(1-\mu_0) + q_B &= \frac{b + a \cdot (1-\mu_0)\delta_B/2 + a\mu_0 - d}{\delta_B/2}.
\end{aligned}$$

The expected welfare is given by

$$\begin{aligned}
a \cdot \mu_0(1-\delta) + (a(1-\mu_0) + q_B) \cdot \frac{1-\delta}{2} &= a \cdot \mu_0(1-\delta) + \frac{b + a \cdot (1-\mu_0)\delta_B/2 + a\mu_0 - d}{\delta_B} \cdot (1-\delta) \\
&= \frac{1-\delta}{\delta_B} \cdot \left[a \cdot \left(\frac{\delta_B}{2} + \mu_0 \left(1 + \frac{\delta_B}{2} \right) \right) + b - d \right].
\end{aligned}$$

$\square$

Denote by q_1 the solution to

$$b + (a - q_1) \cdot \delta_S \frac{1}{2} + q_1 \cdot \delta_S \frac{1 + \pi_1}{2} = d \quad \Rightarrow \quad q_1 \stackrel{def}{=} \frac{d - a \cdot \delta_S/2 - b}{\delta_S \pi_1/2} \tag{A.75}$$

where $\pi_1 \stackrel{def}{=} \frac{\mu/2}{\mu/2+1-\mu} = \frac{\mu}{2-\mu}$ is the posterior belief about the asset quality in the failing pool after the first stress test. It follows that $q_1 < a$ whenever $\pi_1| > \pi_{DF}$.

Lemma A.15 (Sequential test dominance.)**.** *Suppose* $\pi_1 \geq \pi_{DF}$ *and* $\min(a - q_1, a) \geq d$*, where* q_1 *is defined by* (A.75)*. Then, the two-step sequential test disclosing half, i.e.,* $\mu/2$*, of the strong banks at the first step, and the second half during the second step achieves the global optimum if*

$$\frac{a(1-\mu) \cdot \delta_S/2 - b \cdot \mu}{\left(\frac{1}{1-\mu/2} - \mu\right) \cdot \delta_S/2} \leq q_1 \leq \frac{b \cdot \mu}{\delta_S}. \tag{A.76}$$

Proof. Consider the following two step sequential test:

Step 1: disclose $\mu/2$ of the strong banks are require the failing banks to sell q_1 quantity of the risky asset

Step 2: disclose the remaining $\mu/2$ of the strong banks and require the weak banks to sell all remaining risky assets

In order for the stress test to have an efficient outcome the passing banks in step 1 should have enough liquidity and balance sheet capacity to buy q_1 of the assets sold by the failing banks. The liquidity constraint is not binding when

$$\begin{aligned} b\frac{\mu}{2} &\geq \left(1 - \frac{\mu}{2}\right) q_1 \delta_S \frac{1 + \pi_1}{2}, \\ b\mu &\geq \left(1 - \frac{\mu}{2}\right) q_1 \delta_S \left(1 + \frac{\mu/2}{1 - \mu/2}\right), \\ b\mu &\geq q_1 \delta_S, \\ \frac{b\mu}{\delta_S} &\geq q_1. \end{aligned}$$

while since $a > a - q_1 > d$, then the strong passing banks' solvency constraint is not binding.

In the second stage the remaining strong banks are disclosed. As a result, both $[0, \mu/2)$ strong banks and $[\mu/2, \mu)$ are buying assets from the weak banks. Since $a > d$ the new purchases do not exhaust the balance sheet capacity of the $[0, \mu/2)$ banks and their available liquidity is $b\frac{\mu}{2} - q_1\frac{\delta_S}{2}$. The default-free constraint of the $[\mu/2, \mu)$ banks is not binding (sufficient condition) when $a - q_1 \geq d$, which is assumed in the formulation of the Lemma. These strong banks have $\frac{\mu}{2}\left(b + q_1\delta_S\frac{1+\pi_1}{2}\right)$ liquidity available. Total liquidity of the strong banks is sufficient to buy the assets of the weak banks when

$$
\begin{aligned}
b\frac{\mu}{2} - q_1\frac{\delta_S}{2} + \frac{\mu}{2}\left(b + q_1 \cdot \delta_S\frac{1+\pi_1}{2}\right) &\geq (1-\mu)(a - q_1) \cdot \frac{\delta_S}{2}, \\
b\frac{\mu}{2} - q_1\frac{\delta_S}{2} + \frac{\mu}{2}\left(b + q_1 \cdot \delta_S\frac{1}{2-\mu}\right) &\geq (1-\mu)(a - q_1) \cdot \frac{\delta_S}{2}, \\
b\mu - q_1\frac{\delta_S}{2}\left(1 - \frac{1}{1-\mu/2}\right) &\geq a \cdot (1-\mu)\frac{\delta_S}{2} - q_1(1-\mu)\frac{\delta_S}{2}, \\
q_1 \cdot \frac{\delta_S}{2}\left(\frac{1}{1-\mu/2} - \mu\right) &\geq a \cdot (1-\mu)\frac{\delta_S}{2} - b\mu.
\end{aligned}
$$

□

Proof of Lemma 5

The $t = 2$ price of the long-term bond in state $\theta = 0$ is given retention A is given by

$$p_2(A, 0, Y; p_1) = \min\left\{p_2^C(A, 0, Y; p_1),\ 1\right\},$$

where

$$p_2^C(A, 0, Y; p_1) \stackrel{def}{=} \frac{b + (a - A) \cdot p_1 - d + \mu_0/2 + (1-\mu_0) \cdot Y}{n - A}$$

is the cash in the market price in period $t = 2$, $p_1 = p_1(A; \pi)$ is the the period $t = 1$ price of the long-term bond given retention A and belief π. It is determined as a solution to the fixed point condition

$$p_1(A; \pi) = \delta \cdot \Big(\pi + (1-\pi) \cdot \mathrm{E}\left[p_2(A, 0, Y; p_1(A; \pi))\right]\Big).$$

That is, the first-period price $p_1(A;\pi)$ is a solution to the fixed point $F(p_1, A; \pi) = 0$, where $F(\cdot, \cdot; \pi)$ is given by

$$F(p_1, A; \pi) \stackrel{def}{=} p_1 - \delta \cdot \Big(\pi + (1-\pi) \cdot \mathrm{E}\left[\min\left\{p_2^C(A, 0, Y; p_1),\ 1\right\}\right]$$
$$= p_1 - \delta \cdot \left(\pi + (1-\pi) \cdot \mathrm{E}\left[\min\left\{\frac{b + (a-A) \cdot p_1 - d + \mu_0/2 + (1-\mu_0) \cdot Y}{n - A},\quad 1\right\}\right]\right).$$

The partial derivative with respect to p_1 is

$$\frac{\partial}{\partial p_1} F(p_1, A; \pi) \geq 1 - \delta(1-\pi)\frac{a-A}{n-A} \geq 1 - \frac{a-A}{n-A} > 0.$$

The derivative with respect to A is given by

$$p_1'(A) = -\frac{\partial F}{\partial A}(p_1, A; \pi) \Big/ \frac{\partial F}{\partial p_1}(p_1, A, \pi) \sim -\frac{\partial F}{\partial A}(\pi_1, A; \pi).$$

The partial derivative w.r.t. A is

$$\begin{aligned}
\frac{\partial}{\partial A} F(p_1, A; \pi) &= -\delta(1-\pi) \cdot \frac{\partial}{\partial A}\mathrm{E}\left[\min\left\{p_2^C(A, 0, Y; p_1), 1\right\}\right] \\
&= -\delta(1-\pi) \cdot \mathrm{E}\left[\frac{\partial}{\partial A} p_2^C(A, 0, Y; p_1) \cdot \mathbb{1}\left\{p_2^C(A, 0, Y; p_1) < 1\right\}\right] \\
&= -\delta(1-\pi) \cdot \mathrm{E}\left[\frac{p_2^C(A, 0, Y; p_1) - p_1}{n - A} \cdot \mathbb{1}\left\{p_2^C(A, 0, Y; p_1) < 1\right\}\right] \\
&\sim -\delta(1-\pi) \cdot \mathrm{E}\left[\left(p_2^C(A, 0, Y; p_1) - p_1\right) \cdot \mathbb{1}\left\{p_2^C(A, 0, Y; p_1) < 1\right\}\right] \\
&\sim \mathrm{E}\left[p_1 - p_2^C(A, 0, Y; p_1) \,\middle|\, p_2^C(A, 0, Y; p_1) < 1\right] \\
&= p_1 - \mathrm{E}\left[p_2^C(A, 0, Y; p_1) \,\middle|\, p_2^C(A, 0, Y; p_1) < 1\right]
\end{aligned}$$

In what follows, we provide conditions under which $p_1 \geq \mathrm{E}\left[p_2^C \,\middle|\, p_2^C < 1\right]$. Dropping some notation, obtain

$$\begin{aligned}
p_1 - \mathrm{E}\left[p_2 | p_2 < 1\right] &= \delta\pi + \delta(1-\pi) \cdot \mathrm{E}\left[p_2 | p_2 < 1\right] \cdot \mathrm{P}(p_2 < 1) + \delta(1-\pi) \cdot \mathrm{P}(p_2 \geq 1) - \mathrm{E}\left[p_2 | p_2 < 1\right] \\
&= \delta\pi + \delta(1-\pi) \cdot \mathrm{E}\left[p_2 | p_2 < 1\right] \cdot \mathrm{P}(p_2 < 1) + \delta(1-\pi) \cdot (1 - \mathrm{P}(p_2 < 1)) - \mathrm{E}\left[p_2 | p_2 < 1\right]
\end{aligned}$$

Denote $y = \mathrm{P}(p_2 < 1)$. As $Y \sim U[0,1]$, it solves

$$\frac{b - d + (a - A) \cdot p_1 + \mu_0/2 + (1-\mu_0) \cdot y/(1-\pi)}{n - A} = 1,$$
$$\frac{b - d + (a - A) \cdot p_1 + \mu_0/2}{n - A} = 1 - \frac{1-\mu_0}{n-A} \cdot \frac{y}{1-\pi}.$$

Then

$$\mathrm{E}\left[p_2 | p_2 < 1\right] = \frac{1}{2}\left(1 - \frac{1-\mu_0}{n-A} \cdot \frac{y}{1-\pi} + 1\right) = 1 - \frac{1}{2} \cdot \frac{1-\mu_0}{n-A} \cdot \frac{y}{1-\pi}.$$

Substituting into the above condition obtain

$$\begin{aligned}
\frac{\partial}{\partial A} F(p_1, A; \pi) &\sim \delta\pi + \delta(1-\pi)\left(1 - \frac{1}{2}\frac{1-\mu_0}{n-A} \cdot \frac{y}{1-\pi}\right) y + \delta(1-\pi)(1-y) - \left(1 - \frac{1}{2} \cdot \frac{1-\mu_0}{n-A}\frac{y}{1-\pi}\right) \\
&= \delta - \delta \cdot \frac{1}{2} \cdot \frac{1-\mu_0}{n-A} \cdot y^2 - 1 + \frac{1}{2} \cdot \frac{1-\mu_0}{n-A} \cdot \frac{y}{1-\pi} \\
&= \delta - 1 + \frac{1}{2} \cdot \frac{1-\mu_0}{n-A} \cdot y \left(\frac{1}{1-\pi} - \delta \cdot y\right)
\end{aligned}$$

The above expression is increasing in δ and y, as can be seen from

$$\frac{\partial}{\partial \delta}\frac{\partial}{\partial A} F(p_1, A; \pi) = 1 - \frac{y^2}{2} \cdot \frac{1-\mu_0}{n-A} \geq 1 - \frac{1}{2} \cdot \frac{1-\mu_0}{n-A} \cdot \frac{y}{1-\pi} \geq 0.$$
$$\frac{\partial}{\partial y}\frac{\partial}{\partial A} F(p_1, A; \pi) = \frac{1}{2} \cdot \frac{1-\mu_0}{n-A} \cdot \left(\frac{1}{1-\pi} - 2\delta \cdot y\right) \geq 0, \tag{A.77}$$

where (A.77) follows from

$$\frac{b - d + (a - A) \cdot p_1 + \mu_1/2}{n - A} \geq 1 = \frac{b - d + (a - A) \cdot p_1 + \mu_0/2 + (1-\mu_0) \cdot y/(1-\pi)}{n - A}$$
$$\mu_1/2 \geq \mu_0/2 + (1-\mu_0) \cdot \frac{y}{1-\pi}$$
$$\frac{\mu_1 - \mu_0}{2(1-\mu_0)} \geq \frac{y}{1-\pi}$$
$$y \leq (1-\pi) \cdot \frac{\mu_1 - \mu_0}{2(1-\mu_0)} \leq \frac{1}{2}.$$

A.9 Proof of Proposition 5

Consider the bank's portfolio (b, a) and a fixed fraction of aggregate risk is μ. The price of the risky asset is

$$n \cdot p_2(\hat{a}, \theta, Y) = b + (a - \hat{a}) \cdot p_1(\hat{a}) + \hat{a} \cdot p_2(\hat{a}, \theta, Y) + \mu_\theta/2 + (1 - \mu_\theta) \cdot Y - d$$

$$p_2(\hat{a}, \theta, Y) = \min\left\{\frac{b + (a - \hat{a}) \cdot p_1(\hat{a}) - d + \mu_\theta/2 + (1 - \mu_\theta) \cdot Y}{n - \hat{a}}, \quad 1\right\}. \tag{A.78}$$

The price of the risky asset at $t = 1$ is pinned down by the expected resale value at $t = 2$ given by

$$p_1(\hat{a}) = \delta \cdot \mathrm{E}\left[p_2(\hat{a}, \theta, Y)\right] = \mathrm{E}\left[\min\left\{\frac{b + (a - \hat{a}) \cdot p_1(\hat{a}) - d + \mu_\theta/2 + (1 - \mu_\theta) \cdot Y}{n - \hat{a}}, \quad 1\right\}\right]. \tag{A.79}$$

The default-free constraint is given by

$$b + (a - \hat{a}) \cdot p_1(\hat{a}) + \hat{a} \cdot p_2(\hat{a}, \theta, Y) \overset{P\text{-}a.s.}{\geq} d. \tag{A.80}$$

As can be seen from (A.78), the price of the risky asset is in period $t = 2$ is increasing in the realization of the systematic shock Y, implying that it is sufficient for (A.80) to be satisfied at $Y = 0$. Moreover, since $p_2(\hat{a}, \theta, 0)$ is increasing in the fraction of the banks holding idiosyncratic risk μ_θ, it is sufficient for (A.80) to be satisfied for $\theta = 0$. Substituting both $Y = 0$ and $\theta = 0$, (A.80) is equivalent to

$$b + (a - \hat{a}) \cdot p_1(\hat{a}) + \hat{a} \cdot p_2(\hat{a}, 0, 0) \overset{P\text{-}a.s.}{\geq} d. \tag{A.81}$$

Denote by $\bar{a}_{DF} \leq n$ to be the largest solution to the binding constraint (A.81)

$$\bar{a}_{DF} \overset{def}{=} \max\left\{\hat{a} \leq n : \quad b + (a - \hat{a}) \cdot p_1(\hat{a}) + \hat{a} \cdot p_2(\hat{a}, 0, 0) \geq d\right\}. \tag{A.82}$$

Denote by $\bar{a}_B$ to be the maximum quantity of the asset the bank can purchase at $t = 1$ subject to the budget constraint, but ignoring the default-free constraint

$$\bar{a}_B \stackrel{def}{=} \max\Big\{\hat{a} \le n: \qquad \hat{a} \cdot p_1(\hat{a}) \le a \cdot p_1(\hat{a}) + b\Big\}. \tag{A.83}$$

Proposition 5, part 1: optimally binding default-free constraint.

The regulator's objective is to minimize the opportunity cost of capital provided to the banks by the capital market, i.e., to minimize the dollar-value of asset sold. Denote by A the optimal retention quantity given by

$$A \stackrel{def}{=} \arg\min_{\hat{a}}\Big\{(a - \hat{a}) \cdot p_1(\hat{a})\Big\}. \tag{A.84}$$

subject to (A.78), (A.79), (A.80), and (A.83).

Lemma A.16 (Optimally binding default-free constraint)**.** *Suppose $d \ge \mu_0/2$. Then, the optimal asset retention is given by $A = \bar{a}_{DF}$. Consequently, the default-free constraint* (A.80) *is binding if and only if $\bar{a}_{DF} < n$.*

Proof. Suppose that $\bar{a}_{DF} < n$. Then $\bar{a}_{DF}$ satisfies the default-free constraint (A.81) with equality. We now show that asset retention $\bar{a}_{DF}$ satisfies the budget constraint and is, thus, feasible. Substituting $\theta = Y = 0$ into (A.78) and using the fact that the default-free constraint must be binding at $\bar{a}_{DF} < n$ obtain

$$n \cdot p_2(\bar{a}_{DF}, 0, 0) = b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) + \bar{a}_{DF} \cdot p_2(\bar{a}_{DF}, 0, 0) + \frac{\mu_0}{2} - d = \frac{\mu_0}{2}.$$

Substituting $p_2(\bar{a}_{DF}, 0, 0) = \mu_0/2n$ into the binding default-free constraint obtain

$$\begin{aligned}
b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) + \bar{a}_{DF} \cdot p_2(\bar{a}_{DF}, 0, 0) &= d \\
b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) &= d - \bar{a}_{DF} \cdot p_2(\bar{a}_{DF}, 0, 0) \\
b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) &= d - \frac{\bar{a}_{DF}}{n} \cdot \frac{\mu_0}{2} \\
b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) &\stackrel{(i)}{\ge} d - \frac{n}{n} \cdot \frac{\mu_0}{2} \\
b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) &\ge 0
\end{aligned}$$

$$\frac{b}{p_1(\bar{a}_{DF})} + a \geq \bar{a}_{DF} \tag{A.85}$$

where inequality (i) follows from $\bar{a}_{DF} < n$. The resulting comparison (A.85) implies that $\bar{a}_{DF}$ satisfies the budget constraint (A.83) and is, thus, a feasible allocation.

We now show that it must be the case that optimal retention $A = \bar{a}_{DF}$. From the contrary, suppose that $A \neq \bar{a}_{DF}$, i.e., $A < \bar{a}_{DF}$. The optimality of A implies that the dollar-value sale of the risky asset to the market is lower under A than under $\bar{a}_{DF}$, as captured by

$$(a - A) \cdot p_1(A) < (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}). \tag{A.86}$$

Moreover, from (A.86), equation (A.78) implies that

$$\begin{aligned} p_2(A, \theta, Y) &= \min\left[\frac{b + (a - A)p_1(A) + \mu_\theta/2 + (1 - \mu_\theta)Y - d}{n - A}, 1\right] \\ &< \min\left[\frac{b + (a - \bar{a}_{DF})p_1(\bar{a}_{DF}) + \mu_\theta/2 + (1 - \mu_\theta)Y - d}{n - A}, 1\right] \\ &< \min\left[\frac{b + (a - \bar{a}_{DF})p_1(\bar{a}_{DF}) + \mu_\theta/2 + (1 - \mu_\theta)Y - d}{n - \bar{a}_{DF}}, 1\right] = p_2(\bar{a}_{DF}, \theta, Y) \end{aligned} \tag{A.87}$$

Combining (A.86) with (A.87) obtain

$$\begin{aligned} d \leq b + (a - A)p_1(A) + Ap_2(A, 0, 0) &< b + (a - \bar{a}_{DF}) \cdot p_1\left(\bar{a}_{DF}\right) + A \cdot p_2(A, 0, 0) \\ &< b + (a - \bar{a}_{DF}) \cdot p_1\left(\bar{a}_{DF}\right) + Ap_2 \cdot (\bar{a}_{DF}, 0, 0) \\ &< b + (a - \bar{a}_{DF}) \cdot p_1\left(\bar{a}_{DF}\right) + \bar{a}_{DF} \cdot p_2(\bar{a}_{DF}, 0, 0), \end{aligned}$$

which implies that the default-free constraint is slack at $\bar{a}_{DF}$. By continuity, it implies that the default-free constraint (A.81) is also satisfied at $\hat{a} = \bar{a}_{DF} + \varepsilon$ for a small $\varepsilon > 0$, contradicting the definition of $\bar{a}_{DF}$. Hence it must be the case that $A = \bar{a}_{DF}$ whenever $\bar{a}_{DF} < n$.

Next, consider the case $\bar{a}_B < \bar{a}_{DF} = n$. In this case, it follows from (A.78) and (A.79) that $p_2(\bar{a}_{DF}, \theta, Y) = 1$ and $p_1(\bar{a}_{DF}) = \delta$ respectively. Moreover, since the banks hold all the assets, the default-free constraint is

$$b + (a - n) \cdot \delta + \frac{\mu_0}{2} - d \geq 0$$

because idiosyncratic banks have a cash surplus of $\mu_0(b + (a - n)\delta + 0.5 - d)$ and systematic banks have a cash deficit $(1 - \mu_0)(b + (a - n)\delta - d)$, while the 3-period bond is simply used as a means of payment to redistribute cash across banks. Hence

$$b + (a - \bar{a}_{DF}) \cdot p_1(\bar{a}_{DF}) = b + (a - n) \cdot \delta \geq d - \frac{\mu_0}{2} \geq 0,$$

that is, $\bar{a}_{DF} > \bar{a}_B$ satisfies the budget constraint, which is a contradiction with the definition of $\bar{a}_B$ being the largest $\hat{a}$ satisfying A.83. As a result, case $\bar{a}_B < \bar{a}_{DF} = n$ never occurs.

The last case to consider is $\bar{a}_B = \bar{a}_{DF} = n$. Similar to the previous case, it implies that $p_1(n) = \delta$. From the contrary, suppose that $A < n$. Then

$$(a - A) \cdot p_1(A) > (a - n) \cdot p_1(A) \overset{(i)}{\geq} (a - n) \cdot p_1(n), \tag{A.88}$$

where (i) holds because $p_1(A) \leq \delta = p_1(n)$ and $a - n < 0$. Inequality (A.88) implies that retention quantity n, which is feasible, generates greater welfare by distributing capital from the banks to the capital market. As a result, if $\bar{a}_B = \bar{a}_{DF} = n$ then $A = n$ as well. Moreover, in this case the default-free constraint is slack. □

Corollary A.4. *Suppose $d \geq \mu_0/2$. Then*

$$\arg\min_{\hat{a}} \Big\{(a - \hat{a}) \cdot p_1(\hat{a})\Big\} = \arg\max_{\hat{a}} \left\{b + \frac{\mu_0}{2} \cdot \frac{\hat{a}}{n} - d\right\} \tag{A.89}$$

where all optimizations are subject to (A.78)*,* (A.79)*,* (A.83)*, and* (A.80)*. Moreover, if $a_{DF} < n$, then*

$$\min_{\hat{a}} \Big\{(a - \hat{a}) \cdot p_1(\hat{a})\Big\} = \max_{\hat{a}} \left\{b + \frac{\mu_0}{2} \cdot \frac{\hat{a}}{n} - d\right\}.$$

Proof. Since $\hat{a}/n$ is increasing in $\hat{a}$ over $(0, n)$ the solution to the r.h.s. of (A.89) is the maximal $\hat{a}$ that satisfies default-free constraint (A.81) and budget constraint (A.83). Following Lemma A.16

the optimal asset retention is $A = \bar{a}_{DF}$. Since $A = \bar{a}_{DF}$ is the largest $\hat{a}$ satisfying default-free and budget constraints, it maximizes the right hand side f (A.89).

If $A = \bar{a}_{DF} < n$, then the default-free constraint is binding at $\theta = Y = 0$, which can be written as

$$\begin{aligned} b + (a - A) \cdot p_1(A) + A \cdot p_2(A, 0, 0) &= d \\ b + (a - A) \cdot p_1(A) + A \cdot \frac{\mu_0/2}{n} &= d \\ -(a - A) \cdot p_1(A) &= b + \frac{\mu_0}{2} \cdot \frac{A}{n} - d. \end{aligned}$$

□

Lemma A.17 (Equilibrium prices). *Suppose retention quantity A is optimal. The price of the three-year bond at $t = 2$ in state $\theta = 0$ is*

$$p_2(A, 0, Y) = \min\left\{\frac{\mu_0/2}{n} + \frac{(1-\mu_0) \cdot Y}{n - A}, \quad 1\right\}. \tag{A.90}$$

Moreover, if $\bar{a}_{DF} < n$, then $p_2(A, 0, Y) < 1$ for all $Y \in [0, 1]$. In addition, if $\mu_1/2 \geq n$, then $p_2(A, 1, Y) \equiv 1$. The resulting price of the long-term bond at $t = 1$ is

$$p_1(A) = \delta \cdot \left(\pi \cdot 1 + (1 - \pi) \cdot \mathrm{E}\left[\min\left\{\frac{\mu_0/2}{n} + \frac{(1-\mu_0) \cdot Y}{n - A}, \quad 1\right\}\right]\right). \tag{A.91}$$

Proof. Suppose $A = \bar{a}_{DF} = n$. It follows from (A.78) that $p_2(A, \theta, Y) = 1$ and, consequently, and $p_1(A) = \delta$. Hence the result holds if $A = \bar{a}_{DF} = n$.

Suppose $A = \bar{a}_{DF} < n$. The binding default-free constraint at $\theta = Y = 0$ implies that $p_2(A, 0, 0) = \frac{\mu_0/2}{n}$. Whenever $p_2(A, 0, Y) < 1$, (A.78) can be rewritten as

$$\begin{aligned} p_2(A, 0, Y) &= p_2(A, 0, 0) + \frac{1 - \mu_0}{n - A} \cdot Y, \\ p_2(A, 0, Y) &= \frac{\mu_0/2}{n} + \frac{1 - \mu_0}{n - A} Y. \end{aligned}$$

Moreover, $p_2(A, 0, Y)$ is less than 1 whenever the r.h.s. of the above expression is less than 1. Consequently, $p_2(A, 0, Y)$ is given by (A.90).

The market clearing condition in state $\theta = 1$ is given by

$$n \cdot p_2(A, 1, Y) = b + (a - A) \cdot p_1(A) + A \cdot p_2(A, 1, Y) + \mu_1/2 + (1 - \mu_1) \cdot Y - d$$

Two sub-cases are possible.

- Case: $p_2(A, 0, 0) < 1$. Subtracting the market clearing condition for $Y = 0$ in state $\theta = 0$ from state $\theta = 1$ obtain

$$\begin{aligned}
p_2(A, 1, 0) - p_2(A, 0, 0) &= \frac{1}{n - A} \cdot \frac{\mu_1 - \mu_0}{2} \\
p_2(A, 1, 0) &= p_2(A, 0, 0) + \frac{1}{n - A} \cdot \frac{\mu_1 - \mu_0}{2} \\
p_2(A, 1, 0) &= \min\left\{p_2(A, 0, 0) + \frac{1}{n - A} \cdot \frac{\mu_1 - \mu_0}{2},\ 1\right\} \\
p_2(A, 1, 0) &= \min\left\{\frac{\mu_0/2}{n} + \frac{1}{n - A} \cdot \frac{\mu_1 - \mu_0}{2},\ 1\right\} \\
p_2(A, 1, 0) &= \min\left\{\frac{n\mu_1 - A\mu_0}{2n(n - A)},\ 1\right\} \\
p_2(A, 1, 0) &\overset{(i)}{\geq} \min\left\{\frac{n\mu_1 - A\mu_1}{2n(n - A)},\ 1\right\} \\
p_2(A, 1, 0) &\geq \min\left\{\frac{\mu_1}{2n},\ 1\right\} \\
p_2(A, 1, 0) &\overset{(ii)}{=} 1
\end{aligned}$$

  where (i) follows from $\mu_1 \geq \mu_0$ and (i) follows from $\mu_1 > 2n$.

- Case: $p_2(A, 0, 0) = 1$. But then $p_2(A, 1, Y) \equiv 1$, and $p_1(A) = \delta$. This also implies that $p_2(A + \varepsilon, 1, Y) \equiv 1$ and $p_1(A + \varepsilon) = \delta$ and $A + \varepsilon$ does not violate thed default-free constraint. This contradicts the fact that $\bar{a}_{DF} < n$.

□

When $\bar{a}_{DF} < n$ the optimal asset retention A is pinned down by the default-free condition

$$b + (a - A)\delta\left(\pi + (1 - \pi)\mathrm{E}\left[\min\left\{\frac{\mu_0}{2n} + \frac{(1 - \mu_0)Y}{n - A},\ 1\right\}\right]\right) + A\frac{\mu_0}{2n} = d$$

$$b + (a - A)\delta\left(\pi - 1 + 1 + (1-\pi)\mathrm{E}\left[\min\left\{\frac{\mu_0}{2n} + \frac{(1-\mu_0)Y}{n-A},\ 1\right\}\right]\right) + A\frac{\mu_0}{2n} = d$$

$$(1-\pi)\left(\mathrm{E}\left[\min\left\{\frac{\mu_0}{2n} + \frac{(1-\mu_0)Y}{n-A},\ 1\right\}\right] - 1\right) \overset{(i)}{=} \frac{d - (a-A)\delta - b - A\frac{\mu_0}{2n}}{\delta(a-A)}$$

It is useful to interpret equality (i) as pinning down the minimal belief $\pi(A)$ at which the bank is able to retain asset quantity A while remaining safe in period $t = 2$:

$$\begin{aligned}
1 - \pi(A) &\overset{def}{=} \frac{b + (a-A)\delta - d + Ap_2(0)}{\delta(a-A)\left(1 - \mathrm{E}\left[\min\left\{p_2(0) + \frac{(1-\mu_0)Y}{n-A},\ 1\right\}\right]\right)} \\
&= \frac{b + (a-A)\delta - d + Ap_2(0)}{\delta(a-A)\cdot\frac{(1-p_2(0))^2}{2}\cdot\frac{n-A}{1-\mu_0}} \\
&= 2\frac{1-\mu_0}{\delta(1-p_2(0))^2}\cdot\frac{b + (a-A)\delta - d + Ap_2(0)}{(a-A)(n-A)} \\
&= 2\frac{1-\mu_0}{\delta(1-p_2(0))^2}\cdot\left(\frac{b + a\cdot\mu_0/2n - d}{(a-A)(n-A)} + \frac{\delta - \mu_0/2n}{n-A}\right)
\end{aligned} \tag{A.92}$$

where $p_2(0) = p_2(A, 0, 0) = \frac{\mu_0/2}{n}$. It is convenient to identify the domain of $\pi(A)$. If $\pi = 1-$, then, from (A.91) it follows that $p_1(A(1-)) = \delta$. The binding default-free condition then implies that the optimal retention quantity $A(1-)$ when belief π is close to 1 is given by

$$b + (a - A(1-))\cdot\delta - d + A(1-)\cdot\frac{\mu_0/2}{n} = 0,$$

$$A(1-) = \frac{b + a\delta - d}{\delta - \mu_0/2n}.$$

This means that for $\pi < 1$ the optimal asset retention $A(\pi) \in [0, \bar{A})$, where $\bar{A} \overset{def}{=} A(1-)$.

Define $\pi_{DF} = \pi(0)$ as the minimal belief at which the bank is solvent if it retains no risky asset. Following (A.92) it solves

$$1 - \pi_{DF} \overset{def}{=} 1 - \pi(0) = 2\frac{1-\mu_0}{\delta(1-p_2(0))^2}\cdot\frac{b + a\delta - d}{a\cdot n}. \tag{A.93}$$

Define by $\alpha(\pi)$ the maximum probability of $\theta = 1$ states being revealed while subject to the banks being able to recapitalize safely by selling *all* of their risky assets, if $\theta = 1$ is not revealed. The latter condition requires that, conditional on non-disclosure of the $\theta = 1$ state, the posterior belief is equal to π_{DF} defined in (A.93). It then follows from the martingale property that

$$
\begin{aligned}
\alpha(\pi) \cdot 1 + (1-\alpha(\pi)) \cdot \pi_{DF} = \pi \\
\alpha(\pi) &= \frac{\pi - \pi_{DF}}{1-\pi_{DF}} = 1 - \frac{1-\pi}{1-\pi_{DF}} \\
1 - \alpha(\pi(A)) &= \frac{1-\pi(A)}{1-\pi(0)}
\end{aligned}
$$

Proposition 5, part 2: optimality of the pass-fail static stress test.

In what follows, we provide sufficient conditions for the pass-fail test with posterior beliefs in $\{\pi_{DF}, 1\}$ to dominate no information for any $\pi \in [\pi_{DF}, 1]$.

Lemma A.18 (Sufficient condition for adverse pass-fail test.)**.** *Suppose $\bar{a}_{DF} < n$ and $\mu_1/2 \geq n$. The adverse pass-fail test dominates no information as long as*

$$
\frac{\mu_0/2}{n} \cdot \frac{b + a\delta}{(b + a\delta - d)d} + \frac{n+a}{na} \leq \frac{\delta}{b + a\delta - d} \tag{A.94}
$$

which, for $\mu_0 = 0$ simplifies to $b + a\delta/2 - d < 0$.

Proof. The expected cost of capital under no information is weakly higher than under the pass-fail stress test

$$
\begin{aligned}
\overbrace{d - b - \frac{\mu_0}{2} \cdot \frac{A}{n}}^{\substack{\text{cost of capital} \\ \text{no information}}} &\geq \alpha(\pi(A)) \cdot \overbrace{(-b)}^{\substack{\text{bank buys long-term} \\ \text{bond from market}}} + \Big(1 - \alpha(\pi(A))\Big) \cdot \overbrace{(d-b)}^{\substack{\text{cost of capital} \\ \text{full sale}}} \\
d - Ap_2(0) &\geq \Big(1 - \alpha(\pi(A))\Big) \cdot d \\
d - Ap_2(0) &\geq \frac{1-\pi(A)}{1-\pi(0)} \cdot d \\
d - Ap_2(0) &\geq \left(1 - A\frac{\delta - p_2(0)}{b + a\delta - d}\right) \cdot \frac{a \cdot n}{(a-A)(n-A)} \cdot d
\end{aligned}
$$

Notice that A is bounded above by $\bar{A} \stackrel{def}{=} \frac{b+a\delta-d}{\delta-p_2(0)}$ because the most lax default-free constraint is at

$p_1(1-) = \delta$. Hence, we need to show that for all $A \in [0, \bar{A}]$

$$d \geq f(A) \overset{def}{=} Ap_2(0) + \left(1 - \frac{A}{\bar{A}}\right) \cdot \frac{a \cdot n}{(a-A)(n-A)} \cdot d$$

Notice that $f(0) = d$, hence the inequality holds for $A = 0$. Next check the derivative

$$\begin{aligned}
f'(A) &= p_2(0) + d \cdot \left[-\frac{1}{\bar{A}} \cdot \frac{a \cdot n}{(a-A)(n-A)} - \left(1 - \frac{A}{\bar{A}}\right) \cdot \frac{a \cdot n}{(a-A)^2(n-A)^2}(2A - n - a) \right] \\
&= p_2(0) + d \cdot \frac{an}{(n-A)(a-A)} \cdot \left[-\frac{1}{\bar{A}} + \left(1 - \frac{A}{\bar{A}}\right) \cdot \frac{n + a - 2A}{(a-A)(n-A)} \right] \qquad \text{(A.95)} \\
&= p_2(0) + d \cdot \frac{an}{(n-A)(a-A)} \cdot \left[-\frac{1}{\bar{A}} + \left(1 - \frac{A}{\bar{A}}\right) \cdot \left(\frac{1}{n-A} + \frac{1}{a-A} \right) \right] \\
&= p_2(0) + d \cdot \frac{an}{(n-A)(a-A)} \cdot \left[-\frac{1}{\bar{A}} + \frac{1}{\bar{A}} \cdot \left(\frac{\bar{A} - A}{n-A} + \frac{\bar{A} - A}{a-A} \right) \right] \\
&\leq p_2(0) + d \cdot \frac{an}{(n-A)(a-A)} \cdot \left[-\frac{1}{\bar{A}} + \frac{1}{\bar{A}} \cdot \left(\frac{\bar{A}}{n} + \frac{\bar{A}}{a} \right) \right] \\
&= p_2(0) + d \cdot \frac{an}{(n-A)(a-A)} \cdot \left[-\frac{1}{\bar{A}} + \frac{1}{n} + \frac{1}{a} \right] \qquad \text{(A.96)}
\end{aligned}$$

In order for $d \geq f(A)$ for $A \in [0, \bar{A}]$ it is sufficient to have $f'(A) \leq 0$. From (A.96) it follows that $f'(A)$ is increasing in A. It is, then, sufficient to check that $f'(0)) \leq 0$, captured by

$$\begin{aligned}
-\frac{1}{\bar{A}} + \frac{1}{n} + \frac{1}{a} &\leq -\frac{p_2(0)}{d} \\
\frac{p_2(0)}{d} + \frac{1}{n} + \frac{1}{a} &\leq \frac{\delta - p_2(0)}{b + a\delta - d} \\
p_2(0) \cdot \left(\frac{1}{d} + \frac{1}{b + a\delta - d} \right) + \frac{1}{n} + \frac{1}{a} &\leq \frac{\delta}{b + a\delta - d} \\
p_2(0) \cdot \frac{b + a\delta}{(b + a\delta - d)d} + \frac{n+a}{na} &\leq \frac{\delta}{b + a\delta - d} \\
\frac{\mu_0/2}{n} \cdot \frac{b + a\delta}{(b + a\delta - d)d} + \frac{n+a}{na} &\leq \frac{\delta}{b + a\delta - d}. \qquad \text{(A.97)}
\end{aligned}$$

If $\mu_0 = 0$ we have $p_2(0) = \frac{\mu_0/2}{n} = 0$ and inequality (A.97) simplifies to

$$\begin{aligned}
\frac{n+a}{na} &\leq \frac{\delta}{b + a\delta - d}, \\
b + a\delta - d &\leq a\delta \frac{n}{n+a}.
\end{aligned}$$

Since $n \geq a$ we have $n/(n+a) \geq 1/2$, it follows that

$$b + a\delta - d \overset{(i)}{<} a\delta\frac{1}{2} \leq a\delta\frac{n}{n+a}$$

where inequality (i) is equivalent to $b + a\delta\frac{1}{2} - d < 0$. □

Proposition 5, part 2: optimality of precautionary recapitalization.

Lemma A.19 (Benefits of Precautionary recapitalization)**.** *Suppose $\pi_{DF} > 0$. Precautionary recapitalization is welfare improving as long as there is a possibility of fire sales in state $\theta = 0$.*

Proof. Substituting $A = 0$ obtain that

$$\begin{aligned} 1 - \pi_{DF} &= 2 \cdot \frac{1-\mu_0}{\delta(1-\mu_0/2n))^2} \cdot \left(\frac{b + a \cdot \mu_0/2n - d}{an} + \frac{\delta - \mu_0/2n}{n}\right) \\ &= 2 \cdot \frac{1-\mu_0}{\delta(1-\mu_0/2n))^2} \cdot \frac{b + a \cdot \delta - d}{an}. \end{aligned}$$

The expected cost of capital from the static stress test is

$$\begin{aligned} \frac{\pi - \pi_{DF}}{1-\pi_{DF}} \cdot (-b) + \frac{1-\pi}{1-\pi_{DF}} \cdot (d-b) &= -b + \frac{1-\pi}{1-\pi_{DF}} \cdot d \\ &= -b + \frac{\delta}{2}\frac{(1-\mu_0/2n)^2}{1-\mu_0} \cdot \frac{an}{b + a\delta - d}. \end{aligned}$$

The price of the risky asset prior to the optimal test

$$p_1(\pi) = \delta \cdot \pi + \delta \cdot (1-\pi)\mathrm{E}\left[\min\left(\frac{\mu_0}{2n} + \frac{(1-\mu_0)Y}{n}, \quad 1\right)\right].$$

Note that the price under the optimal subgame stress test is unaffected by the bank's portfolio. Consider a marginal pre-sale of size ε. The expected efficiency changes the bank's starting portfolio to

$$\hat{b} = b + \varepsilon \cdot p_1(\pi), \qquad \hat{a} = a - \varepsilon.$$

The expected welfare is

$$
\begin{aligned}
&-b+\frac{\delta}{2}\frac{(1-\mu_0/2n)^2}{1-\mu_0}\cdot\frac{\hat{a}n}{\hat{b}+\hat{a}\delta-d} \\
=&-b-\varepsilon\cdot p_1(\pi)+\varepsilon\cdot p_1(\pi)+\frac{\delta}{2}\frac{(1-\mu_0/2n)^2}{1-\mu_0}\cdot\frac{(a-\varepsilon)n}{b+\varepsilon\cdot p_1(\pi)+(a-\varepsilon)\delta-d} \\
=&-b+\frac{\delta}{2}\frac{(1-\mu_0/2n)^2}{1-\mu_0}\cdot\frac{(a-\varepsilon)n}{b+\varepsilon\cdot p_1(\pi)+(a-\varepsilon)\delta-d}.
\end{aligned}
$$

The derivative with respect to ε, scaled by a constant and evaluated at $\varepsilon=0$, is

$$
\begin{aligned}
-\frac{n}{b+a\delta-d}+\frac{an(\delta-p_1(\pi))}{(b+a\delta-d)^2} &\vee 0, \\
-(b+a\delta-d)+a(\delta-p_1(\pi)) &\vee 0, \\
d-b-a\cdot p_1(\pi) &\overset{<}{\vee} 0,
\end{aligned}
$$

where the last inequality holds because we assume that $\pi>\pi_{DF}$. □

A.10 Auxiliary analysis: portfolio choice under a linear distress cost

Lemma A.20 (Optimal portfolio choice under linear distress cost)**.** *Suppose the distress cost function $H(x) = H'(0) \cdot \min(x, 0)$ is linear in the bank's capital shortfall. Then, the optimal asset retention set by the regulator is*

$$A(\pi) = \min\left\{ \max\left\{ \frac{\left|b + a \cdot \delta\frac{1+\pi}{2} - d\right| \times \sqrt{H'(0) \cdot \frac{1-\pi}{1+\pi}}}{\sqrt{H'(0) \cdot \delta^2 \cdot \frac{1-\pi^2}{4}} - (1-\delta)}, \quad \frac{d - b - a\delta\frac{1+\pi}{2}}{1 - \delta\frac{1+\pi}{2}} \right\}, \quad \frac{b + a \cdot \delta\frac{1+\pi}{2}}{\delta\frac{1+\pi}{2}} \right\}. \tag{A.98}$$

Proof. Just like in Section 3, denote by $p \stackrel{def}{=} \delta\frac{1+\pi}{2}$ to be the price of the risky asset, and $w \stackrel{def}{=} b + a \cdot p = b + a \cdot \delta\frac{1+\pi}{2}$ to be the market value of the bank's assets given belief π. The regulator's payoff can be written as

$$\max_{\hat{a}} \left\{ (a - \hat{a}) \cdot \delta\frac{1+\pi}{2} + \hat{a} \cdot \frac{1+\pi}{2} + \mathrm{E}\left[H\Big(w - d + \hat{a} \cdot \cdot (X - p) \Big) \right] \right\}.$$

It is without loss to restrict attention to portfolios in which the low type defaults with positive probability. $w + \hat{a} \cdot (0 - p) \leq d$. Given asset holdings $\hat{a}$, the expected distress cost in state $\theta = 0$ can be written as

$$\begin{aligned} \mathrm{E}_0\left[H\Big(w - d + \hat{a}(X - p) \Big) \right] = \frac{H'(0)}{2\hat{a}} &\times \Big(\min\big[w - d + \hat{a}(1-p), 0 \big] - \min\big[w - d + \hat{a}(0-p), 0 \big] \Big) \\ &\times \Big(\min\big[w - d + \hat{a}(1-p), 0 \big] + \min\big[w - d + \hat{a}(0-p), 0 \big] \Big). \end{aligned} \tag{A.99}$$

The expected distress cost in state $\theta = 1$ is simply $H\left(w - d + \hat{a}(1-p)\right)$.

Case 1: $\frac{w}{p} \leq d$. Then, even if the bank buys the maximum quantity of the asset, $\hat{a} = \frac{w}{p}$, then it still defaults with certainty as can be seen from

$$w - d + \frac{w}{p} \cdot (1 - p) = \frac{w}{p} - d \leq 0. \tag{A.100}$$

In this case, the bank defaults with certainty and, due to linearity of the regulator's objective in this region, it is optimal to set $\hat{a} = \frac{w}{p} = \frac{w}{\delta\frac{1+\pi}{2}}$.

Case 2: $\frac{w}{p} \geq d$**.** Suppose for the optimal $\hat{a}$ it were the case that the bank defaulted conditional on $X = 1$, thus, with certainty. Then, due to the linearity of the regulator's distress cost, it would lead to a higher $\hat{a}$. This implies that if $\frac{w}{p} \geq d$, then the optimal $\hat{a}$ must lead the bank to be solvent if $X = 1$, i.e.,

$$w - d + \hat{a} \cdot (1 - p) \geq 0 \qquad \Leftrightarrow \qquad \hat{a} \geq \frac{d - w}{1 - p} = \frac{d - w}{1 - \delta\frac{1+\pi}{2}}. \tag{A.101}$$

This implies that the optimal $\hat{a}$ must satisfy $w - d + \hat{a} \cdot (0 - p) \leq 0 \leq w - d + \hat{a} \cdot (1 - p)$. Rewrite the expected distress cost (A.99) as

$$\begin{aligned} \mathrm{E}_0\left[H\left(w - d + \hat{a}(X - p)\right)\right] &= -\frac{H'(0)}{2\hat{a}} \cdot \Big(w - d + \hat{a} \cdot (0 - p)\Big) \cdot \Big(w - d + \hat{a} \cdot (0 - p)\Big) \\ &= -\frac{H'(0)}{2\hat{a}} \cdot \Big(w - d - \hat{a}p\Big)^2. \end{aligned}$$

The high type is solvent with certainty, implying that the regulator's objective can be written as

$$\hat{a} \cdot (1 - \delta)\frac{1 + \pi}{2} - \frac{H'(0)(1 - \pi)}{2\hat{a}} \cdot \Big(w - d - \hat{a}p\Big)^2. \tag{A.102}$$

The first order optimality condition with respect to $\hat{a}$ is given by

$$\begin{aligned} (1 - \delta)\frac{1 + \pi}{2} + \frac{H'(0)(1 - \pi)}{2\hat{a}} \cdot \left[\frac{1}{\hat{a}}\Big(w - d - \hat{a}p\Big)^2 + 2p\Big(w - d - \hat{a}p\Big)\right] &= 0, \\ (1 - \delta)\frac{1 + \pi}{2} + \frac{H'(0)(1 - \pi)}{2\hat{a}^2} \cdot \Big(w - d - \hat{a}p\Big)\Big(w - d - \hat{a}p + 2\hat{a}p\Big) &= 0, \\ (1 - \delta)(1 + \pi) + \frac{H'(0)(1 - \pi)}{\hat{a}^2} \cdot \Big(w - d - \hat{a}p\Big)\Big(w - d + \hat{a}p\Big) &= 0, \\ (1 - \delta)(1 + \pi) + \frac{H'(0)(1 - \pi)}{\hat{a}^2} \cdot \Big((w - d)^2 - \hat{a}^2p^2\Big) &= 0, \\ (1 - \delta)(1 + \pi) - H'(0)(1 - \pi)p^2 + \frac{H'(0)(1 - \pi)}{\hat{a}^2} \cdot (w - d)^2 &= 0, \qquad \text{(A.103)} \\ \hat{a}^2 \cdot (1 - \delta)(1 + \pi) + H'(0) \cdot (1 - \pi) \cdot \Big(w - d - \hat{a}p\Big)\Big(w - d + \hat{a}p\Big) &= 0, \\ \hat{a}^2 \cdot \Big((1 - \delta)(1 + \pi) - H'(0) \cdot (1 - \pi) \cdot p^2\Big) + H'(0) \cdot (1 - \pi)(w - d)^2 &= 0, \\ \hat{a}^2 \cdot \left((1 - \delta)(1 + \pi) - H'(0)(1 - \pi) \cdot \delta^2\frac{(1 + \pi)^2}{4}\right) + H'(0) \cdot (1 - \pi)(w - d)^2 &= 0, \\ \hat{a}^2 \cdot \left(1 - \delta - H'(0) \cdot \delta^2\frac{1 - \pi^2}{4}\right) + H'(0) \cdot \frac{1 - \pi}{1 + \pi}(w - d)^2 &= 0. \qquad \text{(A.104)} \end{aligned}$$

The monotonicity of the first-order condition in (A.103) implies[50] that the optimal portfolio is the positive root of quadratic equation (A.103) subject to budget feasibility:

$$\begin{aligned}A(\pi) &= \sup\left\{\hat{a} \leq \frac{w}{p}: \quad \hat{a}^2 \cdot \left(1-\delta - H'(0)\cdot\delta^2\frac{1-\pi^2}{4}\right) + H'(0)\cdot\frac{1-\pi}{1+\pi}\cdot(w-d)^2 \geq 0\right\} \\ &= \min\left[\sqrt{\frac{H'(0)\cdot\frac{1-\pi}{1+\pi}\cdot(w-d)^2}{\max\left(H'(0)\cdot\delta^2\frac{1-\pi^2}{4}-(1-\delta),\ 0\right)}}, \quad \frac{w}{\delta\frac{1+\pi}{2}}\right]\end{aligned}$$

It remains to verify that this optimal portfolio is such that the bank is safe if $X = 1$. This stems from the fact that if $\hat{a}$ is small, the marginal cost changes. In other words, the quadratic nature of the regulator's objective in (A.102) changes. This implies that

$$A(\pi) = \min\left\{\max\left\{\sqrt{\frac{H'(0)\cdot\frac{1-\pi}{1+\pi}\cdot(w-d)^2}{\max\left(H'(0)\cdot\delta^2\frac{1-\pi^2}{4}-(1-\delta),\ 0\right)}}, \quad \frac{d-w}{1-\delta\frac{1+\pi}{2}}\right\}, \quad \frac{w}{\delta\frac{1+\pi}{2}}\right\}.$$

□

Lemma A.21 (Value function upper bound under linear distress cost)**.** *Suppose cost $H(x) = H'(0)\cdot\min(x,0)$ is linear. Suppose $w \leq d$. Then, the regulator's value function is bounded above by*

$$\bar{v}(\pi) = \left(d-b-a\delta\frac{1+\pi}{2}\right)\cdot\phi\left[(1-\delta)\frac{1+\pi}{2} - \frac{1-\pi}{2}H'(0)\left(\delta\frac{1+\pi}{2}+\frac{1}{\phi}\right)^2\right],$$

where $\phi \stackrel{def}{=} \frac{\sqrt{H'(0)\cdot\frac{1-\pi}{1+\pi}}}{\sqrt{H'(0)\cdot\delta^2\cdot\frac{1-\pi^2}{4}-(1-\delta)}}$.

Proof. The value function $V(\pi)$ is dominated by a, possible infeasible, interior solution to the portfolio choice problem if the regulator can borrow at the riskless rate. From Lemma A.20 the optimal portfolio in this relaxed problem is given by

$$\bar{A}(\pi) \stackrel{def}{=} \left(d-b-a\cdot\delta\frac{1+\pi}{2}\right)\times\frac{\sqrt{H'(0)\cdot\frac{1-\pi}{1+\pi}}}{\sqrt{H'(0)\cdot\delta^2\cdot\frac{1-\pi^2}{4}-(1-\delta).}}$$

[50]Formally, the second-order optimality condition is always satisfied.

This implies that the regulator's value function is dominated by

$$v(\pi) \leq \bar{v}(\pi) \overset{def}{=} \bar{A}(\pi)(1-\delta)\frac{1+\pi}{2} + (1-\pi)\mathrm{E}\left[\hat{H}\left(b + a\delta\frac{1+\pi}{2} - d + \bar{A}(\pi)\left(X - \delta\frac{1+\pi}{2}\right)\right)\right]$$

Using the definition of ϕ in the formulation of the Lemma, obtain

$$\begin{aligned}
\bar{v}(\pi) &= (d-w)\phi(1-\delta)\frac{1+\pi}{2} + (1-\pi)\mathrm{E}\left[\hat{H}\left(w - d + (d-w)\phi\left(X - \delta\frac{1+\pi}{2}\right)\right)\right] \\
&= (d-w)\left(\phi(1-\delta)\frac{1+\pi}{2} + (1-\pi)\mathrm{E}\left[\hat{H}\left(\phi\left(X - \delta\frac{1+\pi}{2}\right) - 1\right)\right]\right) \\
&= (d-w)\phi\left((1-\delta)\frac{1+\pi}{2} + (1-\pi)\mathrm{E}\left[\hat{H}\left(X - \delta\frac{1+\pi}{2} - \frac{1}{\phi}\right)\right]\right) \\
&= (d-w)\phi\left[(1-\delta)\frac{1+\pi}{2} - \frac{1-\pi}{2}H'(0)\left(\delta\frac{1+\pi}{2} + \frac{1}{\phi}\right)^2\right] \\
&= \left(d - b - a\delta\frac{1+\pi}{2}\right)\cdot\phi\left[(1-\delta)\frac{1+\pi}{2} - \frac{1-\pi}{2}H'(0)\left(\delta\frac{1+\pi}{2} + \frac{1}{\phi}\right)^2\right]
\end{aligned}$$

□

Corollary A.5 (Optimal portfolio at $\pi = 0$)**.** *Suppose cost $H(\cdot)$ is linear and $H'(0)$ is sufficiently high. Denote $w = b + a \cdot \delta/2$. Then, the optimal portfolio at $\pi = 0$ is*

$$A(0) = \begin{cases} \dfrac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H'(0)}}} & \text{if} \quad d - w \leq 0, \\ \dfrac{d-w}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H'(0)}}} & \text{if} \quad 0 < d - w < w, \\ \dfrac{w}{\delta/2} & \text{if} \quad 0 < w < d - w. \end{cases} \tag{A.105}$$

The expected payoff of this portfolio (for a high enough $H'(0)$) is given by

$$V(0) = \begin{cases} \dfrac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H'(0)}}} \cdot \left[1 - \delta - H'(0) \cdot \dfrac{\delta}{2} \cdot \left(\dfrac{\delta}{2} - \sqrt{\dfrac{\delta^2}{4} - \dfrac{1-\delta}{H'(0)}}\right)\right] & \text{if} \quad d - w \leq 0, \\ \dfrac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H'(0)}}} \cdot \left[-(1-\delta) + H'(0) \cdot \dfrac{\delta}{2} \cdot \left(\dfrac{\delta}{2} + \sqrt{\dfrac{\delta^2}{4} - \dfrac{1-\delta}{H'(0)}}\right)\right] & \text{if} \quad 0 < d - w < w, \\ \dfrac{w}{\delta/2} \cdot \dfrac{1-\delta}{2} - H'(0) \cdot \dfrac{\delta}{4w} \cdot d^2 & \text{if} \quad 0 < w < d - w. \end{cases} \tag{A.106}$$

Proof. At $\pi = 0$ the first term always exceeds the second one in the interior maximum in (A.98), as can be seen from

$$\begin{aligned} \sqrt{\frac{H \cdot (w-d)^2}{\max\left(H \cdot \frac{\delta^2}{4} - (1-\delta),\ 0\right)}} &\gtreqless \frac{d-w}{1-\frac{\delta}{2}} \\ \frac{H \cdot (w-d)^2}{\max\left(H \cdot \frac{\delta^2}{4} - (1-\delta),\ 0\right)} &\gtreqless \frac{(d-w)^2}{\left(1-\frac{\delta}{2}\right)^2} \\ H\left(1-\frac{\delta}{2}\right)^2 &\gtreqless H \cdot \frac{\delta^2}{4} - (1-\delta) \\ H(1-\delta) &\gtreqless -(1-\delta) \end{aligned}$$

Suppose $w > d$. Then the solution is interior and

$$\begin{aligned} V(0) &= A(0) \cdot \frac{1-\delta}{2} - \frac{H}{2A(0)}\left(w - d - A(0) \cdot \frac{\delta}{2}\right)^2 \\ &= \frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \frac{1-\delta}{2} - \frac{H \cdot \sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}}{2(w-d)} \cdot \left(w - d - \frac{(w-d)\frac{\delta}{2}}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}}\right)^2 \\ &= \frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \left[\frac{1-\delta}{2} - \frac{H}{2} \cdot \left(\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}} - \frac{\delta}{2}\right)^2\right] \\ &= \frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \left[\frac{1-\delta}{2} - \frac{H}{2} \cdot \left(\frac{\delta^2}{4} - \frac{1-\delta}{H} + \frac{\delta^2}{4} - \delta \cdot \sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}\right)\right] \\ &= \frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \left[1 - \delta - H \cdot \frac{\delta}{2} \cdot \left(\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}\right)\right] \end{aligned}$$

$$\approx \frac{w-d}{\frac{\delta}{2}} \cdot \frac{1-\delta}{2} = (w-d) \cdot \frac{1-\delta}{\delta} > 0.$$

Suppose $w < d$, but the solution is still interior. Then

$$\begin{aligned}
V(0) &= A(0) \cdot \frac{1-\delta}{2} - \frac{H}{2A(0)} \left(w - d - A \cdot \frac{\delta}{2} \right)^2 \\
&= -\frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \frac{1-\delta}{2} + \frac{H \cdot \sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}}{2(w-d)} \cdot \left(w - d + \frac{(w-d)\frac{\delta}{2}}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \right)^2 \\
&= \frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \left[-\frac{1-\delta}{2} + \frac{H}{2} \cdot \left(\frac{\delta^2}{4} - \frac{1-\delta}{H} + \frac{\delta^2}{4} + \delta \sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}} \right) \right] \\
&= \frac{w-d}{\sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}}} \cdot \left[-(1-\delta) + H \cdot \frac{\delta}{2} \cdot \left(\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \frac{1-\delta}{H}} \right) \right].
\end{aligned}$$

Finally, suppose that $w < d$ and the liquidity constraint is binding. Then

$$V(0) = \frac{w}{\delta/2} \cdot \frac{1-\delta}{2} - \frac{H\delta}{4w} (w - d - w)^2 = \frac{w}{\delta/2} \cdot \frac{1-\delta}{2} - \frac{H\delta}{4w} \cdot d^2 < H \cdot \delta d.$$

□

Corollary A.6 (Different discounts at $\pi = 0$)**.** *Suppose $H(x) = H'(0) \cdot \min(x, 0)$ and $H'(0) \geq \frac{4(1-\delta)}{\delta^2}$. Suppose the bank sells the asset at discount δ_S, but buys the asset back at discount $\delta_B > \delta_S$. Then, the optimal portfolio for $\pi = 0$ is given by*

$$A(0) = \begin{cases} \dfrac{|b + a \cdot \delta_S/2 - d|}{\sqrt{\delta_S^2/4 - (1-\delta)/H'(0)}} & \text{if} \quad \dfrac{(b-d)^2 + a^2(1-\delta)/H'(0)}{a} \leq \delta_S(d-b), \\ \min\left\{ \dfrac{|b + a \cdot \delta_B/2 - d|}{\sqrt{\delta_B^2/4 - (1-\delta)/H'(0)}},\ a + \dfrac{b}{\delta_B/2} \right\} & \text{if} \quad \dfrac{(b-d)^2 + a^2(1-\delta)/H'(0)}{a} \geq \delta_B(d-b), \\ a & \text{otherwise.} \end{cases} \tag{A.107}$$

Proof. It is optimal for the bank to sell some of the asset at discount δ_S if and only if

$$\min\left\{\frac{\left|b+a\cdot\delta_S/2-d\right|}{\sqrt{\delta_S^2/4-(1-\delta)/H'(0)}},\quad \frac{b+a\cdot\delta/2}{\delta/2}\right\}\leq a$$

$$\frac{\left|b+a\cdot\delta_S/2-d\right|}{\sqrt{\delta_S^2/4-(1-\delta)/H'(0)}}\leq a$$

$$\left(b+a\cdot\frac{\delta_S}{2}-d\right)^2\leq a^2\cdot\left(\frac{\delta_S^2}{4}-\frac{1-\delta}{H'(0)}\right)$$

$$(b-d)^2+a\delta_S(b-d)\leq -a^2\cdot\frac{1-\delta}{H'(0)}$$

This necessarily implies that $b<d$ for sales to be ever optimal since the right hand side of the above inequality is negative. Then a sale is optimal if and only

$$\frac{(b-d)^2+a^2\cdot\frac{1-\delta}{H'(0)}}{a(d-b)}\leq\delta_S.$$

Similarly, a purchase is optimal if

$$(b-d)^2+a\cdot\delta_B(b-d)\geq -a^2\cdot\frac{1-\delta}{H'(0)}$$

$$(b-d)^2+a^2\cdot\frac{1-\delta}{H'(0)}\geq a\cdot\delta_B(d-b)$$

Asset purchase is optimal if either $d<b$ or δ_B is not too large, given by the constraint

$$\frac{(b-d)^2+a^2\cdot\frac{1-\delta}{H'(0)}}{a(d-b)}\geq\delta_B.$$

□

B Online Appendix: Extensions and Additional Analysis

B.1 Amendment to Section 5.3: Aggregate and Idiosyncratic Risk

Denote by $\bar{\pi}$ to be

$$\overline{\pi} = -\frac{\left(b + a\delta_S\frac{1}{2} - d\right)(1-\mu_0)}{(\mu_1-\mu_0)a\delta_S\frac{1}{2}}. \tag{B.1}$$

Proposition 6 (Aggregate and Idiosyncratic Risk)**.** *Suppose $\mu_1 > \mu_0$, $\pi > \bar{\pi}$, and $b + a \cdot \mu_0 \geq d$.*

– *The optimal default-free* ***static stress test*** *is a pair of signals $S_a \in \{off, on\}$ and $S_j \in \{pass, fail\}$. The grade $S_a =$ "off$''$ is assigned to the entire system reveals that aggregate state is $\theta = 1$ (risk off), while the second component, S_j is an adverse pass/fail test such that $\mathrm{P}\big(\, X_j = 1 \,\big|\, S_a = off, S_j = fail\big) = \pi_{DF}$. The grade $S_a =$ "on$''$ (risk on) induces low posterior $\mathrm{P}\big(\, \theta = 1 \,\big|\, S_a = on\big) = \bar{\pi} \geq \pi_{DF}$ about the aggregate state and the second component passes only μ_0 of strong banks.*

– *A small* ***precautionary recapitalization*** *is welfare improving if either aggregate prior π sufficiently low or*

$$\frac{b + a\cdot\delta_S(1+\mu_1)/2 - d}{b + a\cdot\delta_S - d} \geq \frac{d-b}{d}.$$

– *The optimal default-free* ***sequential stress test*** *gradually reveals both aggregate and idiosyncratic information and implements the optimal sequential test of the representative bank.*

Proof. Suppose there is a unit mass of banks $j \in [0,1]$. Fraction μ_θ of types have $X_j = 1$ (strong banks) and fraction $1-\mu_\theta$ have $X_j = X \sim U[0,1]$ (weak banks). As before, $\theta \in \{0,1\}$ and $\pi = \mathrm{P}(\theta = 1)$.

Static Default Free Stress Test. To characterize the optimal test we first solve for the optimal idiosyncratic risk disclosure and capital requirements that do not affect the market belief about θ for any prior belief π and then convexity the value function over π.

Suppose that the regulator can disclose α of strong banks, which implies that the posterior about banks that fail the test (are not disclosed to be strong)

$$\mathrm{P}_\pi(X_j = 1 \,|\, Fail) = \frac{\pi\mu_1 + (1-\pi)\mu_0 - \alpha}{1-\alpha}.$$

The maximum quantity of the strong banks to be disclosed without revealing θ is μ_0. The maximum quantity α of disclosure (conditional on π) of strong types subject to not violating the default-free constraint

$$\delta_S \frac{1 + \frac{\pi\mu_1 + (1-\pi)\mu_0 - \alpha}{1-\alpha}}{2} = \frac{d-b}{a}$$

$$\alpha(\pi) = \frac{b + a\delta_S \frac{1+\pi\mu_1+(1-\pi)\mu_0}{2} - d}{b + a\delta_S - d}$$

The threshold level of $\bar{\pi}$ is such that $\alpha(\pi) = \mu_0$ is

$$\frac{\bar{\pi}\mu_1 + (1-\bar{\pi})\mu_0 - \left(\frac{2}{\delta}\cdot\frac{d-b}{a} - 1\right)}{2 - \frac{2}{\delta}\cdot\frac{d-b}{a}} = \mu_0$$

$$\bar{\pi} = -\frac{\left(b + a\delta_S\frac{1}{2} - d\right)(1-\mu_0)}{(\mu_1-\mu_0)a\delta_S\frac{1}{2}}.$$

Note that if π is high, then the above constraint bins at $\alpha = \mu_0$ whenever $\pi < 1$. This implies that at $\pi = 1$ there is a discontinuity.

For low π another constraint might be binding: $\alpha(\pi) = 0$. This pins down the default-free belief level $\pi_{DF}(\mu_0, \mu_1)$

$$0 = \alpha = \frac{\pi_{DF}\mu_1 + (1-\pi_{DF})\mu_0 - \left(\frac{2}{\delta}\cdot\frac{d-b}{a} - 1\right)}{2 - \frac{2}{\delta}\cdot\frac{d-b}{a}}$$

$$0 = \pi_{DF}\mu_1 + (1-\pi_{DF})\mu_0 - \left(\frac{2}{\delta}\cdot\frac{d-b}{a} - 1\right)$$

$$\pi_{DF}(\mu_0,\mu_1) = -\frac{b + a\delta_S\frac{1+\mu_0}{2} - d}{(\mu_1-\mu_0)a\delta_S\frac{1}{2}}.$$

When fraction $\alpha(\pi)$ of the strong banks pass the test they can buy the assets from $1-\alpha(\pi)$ banks that fail the test. There are two types of constraints that the strong banks can face. First is liquidity constraint: strong banks have $b\alpha(\pi)$ cash to purchase assets and the failing banks need to

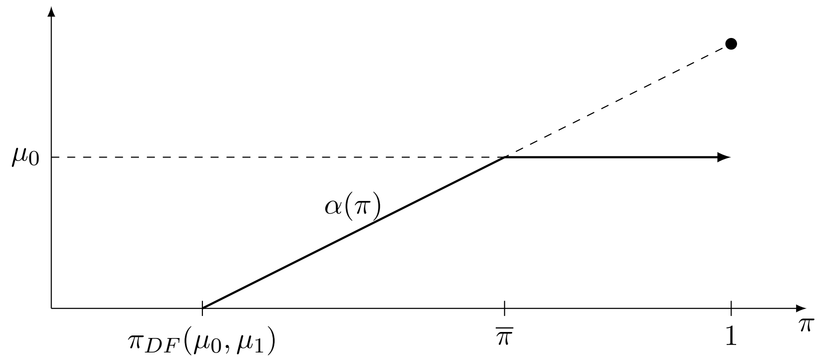

Figure 8: Fraction of strong banks that can be revealed

raise $(1-\alpha(\pi))(d-b)$. Hence, liquidity constraint is binding whenever $\pi < \pi_L$ where

$$\alpha(\pi_L)b = (1-\alpha(\pi_L))(d-b). \qquad \text{(liquidity)}$$

The second type of constraint is solvency. Strong banks are purchasing a mix of assets from the failing banks and with enough bad assets in the mix might become insolvent themselves. When the strong banks buy the assets of the failing banks, they get $\frac{1-\mu_1}{1-\alpha}$ fraction of bad assets, $\frac{\mu_1-\mu_0}{1-\alpha}$ assets that are good with probability π and $\frac{\mu_0-\alpha}{1-\alpha}$ good assets. The price for one unit of this mix is $\frac{d-b}{a}$ (by definition of α). Hence the max quantity q_L that each strong bank can buy subject to remaining default-free is

$$b + q_S\left(\frac{\mu_0-\alpha(\pi)}{1-\alpha(\pi)} - \frac{d-b}{a}\right) + a = d \quad \Rightarrow \quad q_S(\pi) = \frac{b+a-d}{\frac{d-b}{a} - \frac{\mu_0-\alpha(\pi)}{1-\alpha(\pi)}}.$$

The solvency constraint is binding whenever the passing strong banks cannot buy all the assets of the failing banks

$$\alpha(\pi)q_S(\pi) < (1-\alpha(\pi))a \qquad \Leftrightarrow \qquad b + a\mu_0 < d. \qquad \text{(solvency)}$$

Notice whether the solvency constraint binds does not depend on π.

Since by the assumption of the proposition we have $b + a\mu_0 > d$ the solvency constraint never binds, and we need to focus on liquidity constraint only.

First, consider $\pi < \pi_L$. In this case we reveal $\alpha(\pi)$ strong banks who get to keep their assets a

(welfare $a\alpha(\pi)\cdot 1$). Passing strong banks buy $b\cdot\alpha(\pi)$ worth assets from the failing banks and keep those on their balance sheets (welfare $b\alpha(\pi)\frac{a}{d-b}\cdot \mathrm{E}\left[X|fail\right]$). Failing banks sell remaining assets worth $(d-b)(1-\alpha(\pi))-b\alpha(\pi)$ to outsiders (welfare $\delta[(d-b)(1-\alpha(\pi))-b\alpha(\pi)]\cdot\frac{a}{d-b}\mathrm{E}\left[X|fail\right]$. Notice also that the price of the assets of the failing banks is $\delta_S\mathrm{E}\left[X|fail\right]=\frac{d-b}{a}$, hence the overall welfare is

$$
\begin{aligned}
&\alpha a+b\alpha\frac{1}{\delta_S}+\delta[(d-b)(1-\alpha)-b\alpha]\frac{1}{\delta_S}\\
&=\alpha a+(1-\alpha)a\mathrm{E}\left[X|fail\right]-(1-\alpha)a\mathrm{E}\left[X|fail\right]+b\alpha\frac{1}{\delta_S}+\delta[(d-b)(1-\alpha)-b\alpha]\frac{1}{\delta_S}\\
&=a\mathrm{E}\left[X\right]-(1-\alpha)\frac{d-b}{\delta_S}+b\alpha\frac{1}{\delta_S}+\delta[(d-b)(1-\alpha)-b\alpha]\frac{1}{\delta_S}\\
&=a\mathrm{E}\left[X\right]+\frac{1-\delta}{\delta_S}[b\alpha(\pi)-(d-b)(1-\alpha(\pi))]
\end{aligned}
$$

Next, suppose that $\pi>\pi_L$, i.e. the liquidity constraint does not bind. In this case the strong banks have spare $\alpha b-(1-\alpha)(d-b)$ remaining cash that they can use to acquire assets externally. And the overall welfare is

$$
\alpha a+(1-\alpha)a\mathrm{E}\left[X|fail\right]+\frac{1-\delta}{\delta_B}[\alpha b-(1-\alpha)(d-b)] \tag{B.2}
$$

$$
=a\mathrm{E}\left[X\right]+\frac{1-\delta}{\delta_B}[\alpha(\pi)b-(1-\alpha(\pi))(d-b)] \tag{B.3}
$$

Hence, the overall welfare (net of $a\mathrm{E}\left[X\right]$) is a piece-wise linear function of $\alpha(\pi)$ that has a kink at π_L (if $\pi_L<\bar{\pi}$) due to the switch in the directions of the aggregate asset flows.[51] For $\pi<\pi_L$ the banking sector is selling assets to the market, hence the welfare losses are proportional to $\frac{1-\delta}{\delta_S}$ and for $\pi>\pi_L$ the banking sector is buying additional risky assets from the market, hence the welfare gains are proportional to $\frac{1-\delta}{\delta_B}$. The optimal stress test is the a concavification between $\{1,\overline{\pi}\}$.

Precautionary Recapitalization The net effect of precautionary recapitalization for any $\pi<\bar{\pi}$ is positive (with the argument closely following that of Proposition 4) since it allows to disclose more strong banks, i.e., $\alpha(\pi)$ goes up. Moreover, the aggregate informativeness of the stress test

[51]If $\pi_L>\bar{\pi}$ then kink does not occur since $\alpha(\pi)$ in that region is capped at μ_0 and the term $b\alpha-(1-\alpha)(d-b)$ does not change its sign.

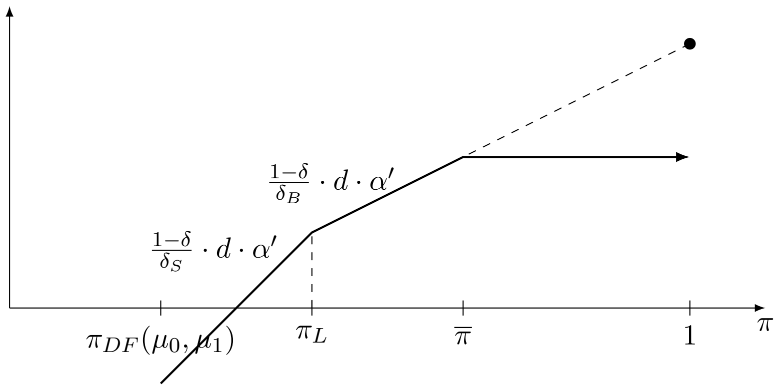

Figure 9: Welfare when solvency constraint does not bind

increases, since $\bar{\pi}$ decreases with marginal precautionary recapitalization.

The only negative effect of precautionary recapitalization could occur when $\theta = 1$. When

$$\frac{b + a \cdot \delta_S (1 + \mu_1)/2 - d}{b + a \cdot \delta_S - d} \geq \frac{d - b}{d}.$$

then even for $\theta = 1$ precautionary recapitalization is beneficial, as shown in Proposition 4. If the above inequality is reversed, then then negative effect of precautionary recapitalization is negligible when π is small, i.e., close to $\pi_{DF}(\mu_0, \mu_1)$.

Sequential Test The optimal sequential test first generates trade across banks and generates a representative bank. Such representative bank has b units of safe asset, $a \cdot \mu_0$ units of rood risky asset, $a(1 - \mu_1)$ units of bad risky assets and $a(\mu_1 - \mu_0)$ risky assets that are either good or bad depending on the state θ. However, since $b + a \cdot \mu_0 > d$ by assumption of the proposition, the representative bank is default free. Hence, after the representative allocation has been achieved the optimal test fully discloses θ and any remaining idiosyncratic information about the banks.

□

B.2 Single Bank Stress Tests under Common Equity Recapitalization

In this section we analyze a version of the model in which the bank improves its capital ratio by raising common equity, rather than selling risky assets. Denote by $\beta < 1$ to be the discount rate shared by all participants in the economy.

Equity capacity. Denote by $\phi \in [0,1]$ to be the fraction of common equity that the bank can sell at any given point. If $\phi = 0$, then the regulator has no capital regulation authority while $\phi = 1$ corresponds to complete dilution of bank shareholders. Since equity recapitalization require shareholder approval, however, it is likely that ϕ is significantly less than 1. Denote by $\tau \in [0,1]$ to be the constant marginal tax rate on the bank shareholders. The capacity of the bank to raise capital is pinned down by the equity issuance constraint

$$\underbrace{e}_{\text{investment}} \leq \underbrace{\phi \cdot (1-\tau) \cdot \beta \cdot \mathrm{E}\left[\max\left\{b + e + a \cdot X - d,\ 0\right\}\right]}_{\text{expected present value of } \phi \text{ shares of the firm}}. \tag{B.4}$$

where we have assumed in (B.4) that bank shareholders have limited liability. Denote by $\bar{e}_I(\pi)$ the bank's equity issuance capacity in the absence of a stress test given belief level π. It is given by the binding solution to (B.4) as

$$\bar{e}_I(\pi) = \beta \cdot (1-\tau) \cdot \phi \cdot \left(\pi \cdot \left[b + \bar{e}_I(\pi) + a \cdot 1 - d\right]^+ + (1-\pi) \cdot \mathrm{E}_0\left[\left[b + \bar{e}_I(\pi) + a \cdot X - d\right]^+\right]\right). \tag{B.5}$$

Lemma B.1. *The bank's equity capacity $\bar{e}_I(\pi)$ is increasing in belief π.*

Proof. The derivative of both sides with respect to (B.5) is given by

$$\begin{aligned}\bar{e}_I'(\pi) = \beta \cdot \phi \cdot (1-\tau) \cdot &\left[\max\left\{b + \bar{e}_I(\pi) + a \cdot 1 - d,\ 0\right\} - \mathrm{E}_0\left[\max\left\{b + \bar{e}_I(\pi) + a \cdot X - d,\ 0\right\}\right]\right] \\ &+ \beta(1-\tau)\phi\Big(\pi \mathbb{1}\left\{b + \bar{e}_I(\pi) + a \cdot 1 - d \geq 0\right\} + (1-\pi)\mathrm{E}_0\left[\mathbb{1}\left\{b + \bar{e}_I(\pi) + a \cdot X - d \geq 0\right\}\right]\Big) \cdot \bar{e}_I'(\pi)\end{aligned}$$

Rearranging terms obtain

$$\bar{e}_I'(\pi) = \frac{\beta\phi(1-\tau) \cdot \left[\max\left\{b + \bar{e}_I(\pi) + a \cdot 1 - d,\ 0\right\} - \mathrm{E}_0\left[\max\left\{b + \bar{e}_I(\pi) + a \cdot X - d,\ \right\}\right]\right]}{1 - \beta(1-\tau) \cdot \phi \mathrm{P}(b + \bar{e}_I(\pi) + a \cdot X - d \geq 0)} \geq 0.$$

□

Denote by π_{DF}^{I} to be the belief threshold at which there is enough equity capacity to avoid bank distress in the worst case outcome $\bar{e}_I\left(\pi_{DF}^{I}\right) = d - b$.

Welfare objective. Denote by $\gamma \in [0, \tau)$ to be the social dead-weight taxation loss on equity payouts.[52] Such social costs imply that it is inefficient to recapitalize the bank just for it to repay all the proceeds to shareholders next period. Such social costs would be warranted if they help mitigate the distress costs. Denote by e_I the size of the equity injection, and e_D as the dividend size paid at $t = 1$. The social welfare is given by

$$\begin{aligned}
\textit{Social Welfare} &\overset{def}{=} \beta \cdot (1 - \gamma \cdot \tau) \left(b + e_I - e_D + a \cdot \frac{1+\pi}{2} - d\right) - e_I + (1 - \gamma \cdot \tau) e_D \\
&\quad - \beta \cdot c\Big(\max\big\{d - b - e_I + e_D - a \cdot X,\ 0\big\}\Big) \\
&= \beta \cdot (1 - \gamma \cdot \tau) \left(b + a\frac{1+\pi}{2} - d\right) - (1 - \beta + \beta\gamma \cdot \tau) \cdot e_I + (1 - \beta)(1 - \gamma \cdot \tau) \cdot e_D \\
&\quad - \beta \cdot c\Big(\max\big\{d - b - e_I + e_D - a \cdot X,\ 0\big\}\Big)
\end{aligned}$$

Optimal Static Stress Test

The marginal social cost of issued capital is given by $\gamma\tau$ which is the counterpart of $1-\delta$ in Sections 2-6. As before, denote $H(x) \overset{def}{=} -\beta \cdot c(-x) \cdot \mathbb{1}\{x \leq 0\}$. The regulator maximizes the expected social welfare, given by

$$\begin{aligned}
V(\pi) = \max_{\hat{e}_I, \hat{e}_D} \Bigg\{ & \beta(1 - \gamma\tau) \left(b + a\frac{1+\pi}{2} - d\right) - \Big(1 - \beta + \beta\gamma\tau\Big)\hat{e}_I \\
& + (1 - \beta)\Big(1 - \gamma\tau\Big)\hat{e}_D + \mathrm{E}\Big[H\Big(b + \hat{e}_I - \hat{e}_D + aX - d\Big)\Big]\Bigg\},
\end{aligned} \tag{B.6}$$

subject to $\hat{e}_I \in [0, \bar{e}_I(\pi)]$ and $\hat{e}_D \geq 0$. Denote by $(e_I(\pi), e_D(\pi))$ the optimal level of recapitalization and dividends given belief π in (B.6). Because of income tax risk it must be the case that either $e_I(\pi) > 0$, or $e_D(\pi) > 0$, but not both.

Lemma B.2 (Optimal Static Stress Test with Equity Issuance)**.** *The optimal static stress test for*

[52] Dividends are taxed at the income tax rate. Feldstein (1999) estimates the marginal dead-weight cost of income tax is 30%, which is quite significant.

multiple banks is characterized by an adverse pass/fail test given by

$$\mathrm{P}(\theta = 1|S = \mathit{pass}) = 1, \qquad \mathrm{P}(\theta = 1|S = \mathit{fail}) = \pi^*,$$

where $\pi^* < \pi_{DF}$ *is such that the bank is in distress with positive probability and the regulator requires the bank to raise* $\bar{e}_I(\pi^*)$ *of capital in exchange for the maximum feasible share* ϕ *of the bank upon receiving the failing grade.*

Proof. Following Lemma A.1, the result of the lemma follows whenever $V'(\pi) \leq \frac{V(1)-V(\pi)}{1-\pi}$ for each π such that $e_I(\pi) \neq \bar{e}(\pi)$.

Applying the Envelope theorem in (B.6) with respect to π obtain

$$V'(\pi) = \beta\Big(1-\gamma\tau\Big)\cdot\frac{a}{2} + H\Big(b+e_I(\pi)-e_D(\pi)+a\cdot 1-d\Big) - \mathrm{E}\Big[H\Big(b+e_I(\pi)-e_D(\pi)+a\cdot X-d\Big)\Big].$$

The star-shaped condition, written as $(1-\pi)\cdot V'(\pi) \leq V(1) - V(\pi)$, can be expressed as

$$\begin{aligned}
&(1-\pi)V'(\pi) - V(1) + V(\pi) \\
=&(1-\pi)\Big[\beta(1-\gamma\tau)a + H\Big(b+e_I(\pi)-e_D(\pi)+a-d\Big) - \mathrm{E}_0\Big[H\Big(b+e_I(\pi)-e_D(\pi)+aX-d\Big)\Big]\Big] \\
&-\beta(1-\gamma\tau)(b+a-d) + \Big(1-\beta+\beta\gamma\tau\Big)e_I(1) - (1-\beta)\Big(1-\gamma\tau\Big)e_D(1) \\
&-H\Big(b+e_I(1)-e_D(1)+a-d\Big) \\
&+\beta(1-\gamma\tau)\cdot\Big(b+a\frac{1+\pi}{2}-d\Big) - \Big(1-\beta+\beta\gamma\tau\Big)e_I(1) + (1-\beta)\Big(1-\gamma\tau\Big)e_D(1) \\
&+\pi\cdot H\Big(b+e_I(\pi)-e_D(\pi)+a-d\Big) + (1-\pi)\cdot\mathrm{E}\Big[H\Big(b+e_I(\pi)-e_D(\pi)+a\cdot X-d\Big)\Big] \\
=&\Big(1-\beta+\beta\gamma\tau\Big)\Big(e_I(1)-e_I(\pi)\Big) - (1-\beta)\Big(1-\gamma\tau\Big)\Big(e_D(1)-e_D(\pi)\Big) \\
&+H\Big(b+e_I(\pi)-e_D(\pi)+a-d\Big) - H\Big(b+e_I(1)-e_D(1)+a-d\Big) \\
=&-\Big[-\Big(1-\beta+\beta\gamma\tau\Big)e_I(1) + (1-\beta)\Big(1-\gamma\tau\Big)e_D(1) + H\Big(b+e_I(1)-e_D(1)+a-d\Big)\Big] \\
&+\Big[-\Big(1-\beta+\beta\gamma\tau\Big)e_I(\pi) + (1-\beta)\Big(1-\gamma\tau\Big)e_D(\pi) + H\Big(b+e_I(\pi)-e_D(\pi)+a-d\Big)\Big] \overset{(i)}{\leq} 0.
\end{aligned}$$

where (i) holds by optimality of $(e_I(1), e_D(1))$ at $\pi = 1$. This implies a contradiction with the fact that $(e_I(\pi), e_D(\pi))$ is an unconstrained optimum. This implies that for every $e_I(\pi) < \bar{e}_I(\pi)$ it follows that $V'(\pi) \leq \frac{V(1)-V(\pi)}{1-\pi}$, implying that, following Lemma A.1, that the regulator prefers to disclose $\theta = 1$ with a small probability rather than not disclosing anything.

Note that for $\pi \geq \pi_{DF}$ it follows that $e_I(\pi) < \bar{e}_I(\pi)$ as, following Lemma B.1, it would imply that for $\pi > \pi_{DF}$, raising the maximum amount of capital makes the bank free of distress, but goes over and raises excess equity which is costly from the tax inefficiency standpoint. Thus, it implies that $e_I(\pi)$ can be equal to $\bar{e}_I(\pi)$ only for $\pi < \pi_{DF}$. □

The payoff from the maximal possible recapitalization

$$v(\pi) \stackrel{def}{=} (1-\gamma) \cdot \left(b + a\frac{1+\pi}{2} - d\right) - \left(1 - \beta + \beta\gamma\tau\right) \cdot \bar{e}(\pi^*) + \mathrm{E}\left[H\left(b + \bar{e}(\pi^*) + aX - d\right)\right].$$

The pooling belief π^* is given by

$$\pi^* = \max\left\{\pi : \quad e_I(\pi) = \bar{e}_I(\pi) \quad \textit{and} \quad v(\pi) = \frac{V(1) - v(\pi)}{1-\pi}\right\}. \tag{B.7}$$

Benefits of Precautionary Recapitalization

Precautionary recapitalization is welfare improving when the ex-ante belief about asset riskiness is sufficiently low.

Lemma B.3 (Optimal Precautionary Recapitalization with Equity Issuance)**.** *Precautionary recapitalization is welfare improving for relatively pessimistic beliefs about asset riskiness, i.e., whenever π is close to π^*.*

Proof. Denote by $V^*(\pi; b, a)$ the regulator's expected payoff from the static test, given safe asset holding b and risky asset holding a. Given that it involves a partially informative test, obtain

$$V^*(\pi; b, a) = \frac{\pi - \pi^*}{1 - \pi^*} \cdot V(1; b, a) + \frac{1 - \pi}{1 - \pi^*} \cdot V(\pi^*; b, a).$$

Consider the effect of the regulator asking the bank to raise ε dollars worth of equity. If $e(1) < \bar{e}(1)$ or $\phi < 1$, then it implies that investors buying the bank's common stock during this precautionary recapitalization will not be perfectly diluted in every state of the world after the stress test is carried out.[53] This implies that if ε is sufficiently low, then the bank can raise this amount, resulting in an objective given by $V^*(\pi; b+\varepsilon, a)-\varepsilon$. Assuming that the expected cost function is sufficiently smooth and applying the Envelope theorem with respect to ε yields that a marginal value of precautionary recapitalization is

$$\begin{aligned}
\frac{\partial V^*}{\partial b}(\pi; b, a) - 1 &= \frac{\pi - \pi^*}{1-\pi^*} \cdot \frac{\partial V^*}{\partial b}(1; b, a) + \frac{1-\pi}{1-\pi^*} \cdot \frac{\partial V^*}{\partial b}(\pi^*; b, a) \\
&= \beta(1-\gamma\tau) - 1 + \frac{\pi - \pi^*}{1-\pi^*} \cdot H'\Big(b + e_I(1) - e_D(1) + a - d\Big) \\
&+ \frac{1-\pi}{1-\pi^*} \cdot \Big[\pi^* \cdot H'\Big(b + \bar{e}(\pi^*) + a - d\Big) + (1-\pi^*) \cdot \mathrm{E}\Big[H'\Big(b + \bar{e}(\pi^*) + aX - d\Big)\Big]\Big].
\end{aligned}$$

Suppose that $\pi = \pi^*$. Then

$$\frac{\partial V^*}{\partial b}(\pi^*; b, a) = -(1-\beta+\beta\gamma\tau) + \pi^* H'\Big(b+\bar{e}(\pi^*)+a-d\Big) + (1-\pi^*)\mathrm{E}_0\Big[H'\Big(b+\bar{e}(\pi^*)+aX-d\Big)\Big] > 0$$

as, at $\pi = \pi^*$ the social value of raising more equity is positive, yet the bank is constrained by its equity capacity. □

The optimal precautionary recapitalization increases the transparency of the stress test via an increase in the bank's equity capacity. If, however, the optimal stress test following precautionary recapitalization does not fully reveal θ, then the regulator can improve by adding a full disclosure step and, potentially, relaxing the capital requirement on the bank in state $\theta = 1$. This allows the bank to pay shareholders dividends on period earlier, economizing on the discount factor β.

Corollary B.1. *The sequential stress test is welfare-improving whenever the optimal precautionary recapitalization does not yield a fully informative test. Any optimal sequential stress test without loss eventually reveals θ fully.*

[53] It is true that subsequent dilution affects the pricing of equity, and the degree of dilution of existing shareholders. As that in itself is just a transfer from old to new shareholders, it does not affect the regulator's objective as she cares only about the dollar amount raised.

Proof. Suppose, following optimal precautionary recapitalization, the optimal stress test is not fully informative. Then, by fully disclosing θ, the regulator can allow the bank to pay back some of its cash holdings as dividends, thus speeding up consumption. Such actions also increase the bank's equity capacity as bank shareholders expect to be paid out sooner, which further increases the set of tests available. □

B.3 Idiosyncratic versus Aggregate Risk Numerical Examples

In this section we provide the constrained static test maximization used to obtain the optimal precautionary recapitalization in Figure 7. When testing multiple banks, the regulator accounts for the possibility of interbank trade. When testing a single bank, the regulator does not internalize the possibility of interbank trade. Formulating the regulator's optimization problems as linear programs provides a simple way for numerical comparisons of optimal precautionary recapitalization policies depicted in Figure 7. The accompanying code is available upon request.

Static Test Program with Multiple Banks

The static test has a passing rate of $\alpha = \frac{\mu-\mu_{DF}}{1-\mu_{DF}}$. In order to account for the possibility of high quality asset outstanding in the market due to sales by the banks in the previous period, assume there is a quantity Q of the high quality asset outstanding in the capital market. One needs to think of Q as a state variable, that tracks the quantity of high quality risky asset in the market. The approach to handle all of the constraints and trading frictions is to formulate the allocation problem as a linear problem subject to the numerous constraints. Denote by

- q_I the quantity of the asset each passing bank purchases from the failing bank;
- q_1 the quantity of the high quality asset each passing bank purchases from the market;
- q_0 the quantity of the low quality asset each passing bank purchases from the market.

The regulator's welfare maximization is given by the linear program in quantities q_0, q_1, and q_I:

$$\max_{\{q_I,q_1,q_0\}} \left\{ \alpha \cdot (1-\delta) \cdot \left[a \cdot 1 + q_I \cdot \frac{1+\mu_{DF}}{2} + q_1 \cdot 1 + q_0 \cdot \frac{1}{2} \right] \right\} \tag{B.8}$$

subject to

$$\begin{cases} b - q_I \cdot \delta_S \dfrac{1+\mu_{DF}}{2} - q_1 \cdot \delta_B - q_0 \cdot \dfrac{\delta_B}{2} \geq 0, & \textit{(liquidity)} \\ a + q_I \cdot \mu_{DF} + q_1 + b - q_I \cdot \delta_S \dfrac{1+\mu_{DF}}{2} - q_1 \cdot \delta_B - q_0 \cdot \dfrac{\delta_B}{2} \geq d, & \textit{(solvency)} \\ 0 \leq \alpha \cdot q_I \leq (1-\alpha) \cdot a \\ 0 \leq \alpha \cdot q_1 \leq Q \\ 0 \leq q_0. \end{cases} \tag{B.9}$$

The payoff in (B.8) subject to (B.9) corresponds to the expected payoff to the regulator in a static test. We use this as the optimal continuation value to the regulator's decision to precautionary recapitalize the banks one step before. We plot the magnitude of this optimal recapitalization in Figure 7 where we denote it by $q_{with\ interbank}(\cdot)$. The python code for this, and all other plots in the paper is available upon request.

Static Test Program with a Single Bank

The static test has a pass rate of $\alpha = \frac{\mu - \mu_{DF}}{1-\mu_{DF}}$. In order to account for the possibility of high quality asset outstanding in the market due to sales by the banks in the previous period, assume there is a quantity Q of the high quality asset outstanding in the capital market. As, in this counterfactual experiment, the regulator does not account for the possibility of interbank trade, the regulator only chooses

- q_1 as the quantity of the high quality asset each passing bank purchases from the market;
- q_0 as the quantity of the low quality asset each passing bank purchases from the market.

The linear optimization is given by

$$\max_{\{q_1,q_0\}} \left\{ (1-\delta) \cdot \left(a \cdot 1 + q_1 \cdot 1 + q_0 \cdot \frac{1}{2} \right) \right\} \tag{B.10}$$

subject to

$$
\begin{cases}
b - q_1 \cdot \delta_B - q_0 \cdot \dfrac{\delta_B}{2} \geq 0, & \textit{(liquidity)} \\
a + q_I \cdot \mu_{DF} + q_1 + b - q_1 \cdot \delta_B - q_0 \cdot \dfrac{\delta_B}{2} \geq d, & \textit{(solvency)} \\
0 \leq q_1 \leq Q \\
0 \leq q_0
\end{cases}
\tag{B.11}
$$

The (nearly inconsequential) difference between (B.10) subject to (B.11) and the optimal stress test obtained in Proposition 1 manifests itself in the case that there is not an infinite quantity of good quality asset outstanding. This distinction bears no consequences for most parameters as, for the strong bank, the liquidity constraint binds more frequently than the solvency constraint. We go through the motions of setting up (B.10) is to have a numerical check that this is the case for the range of parameters tested. This way, the solution to (B.10) corresponds to the hypothetical continuation value to the regulator's decision to precautionary recapitalize the banks one step before. The subtlety is that, in this example, the regulator does not internalize the possibility for efficient reallocation across banks, making this policy suboptimal. We plot the magnitude of this optimal recapitalization in Figure 7 where we denote it by $q_{without\ interbank}(\cdot)$. All python code is available upon request.